Erasure and Duplication in Classical Computations

Pierre Lescanne Dragiša Žunić

JFLA Aix-les-bains
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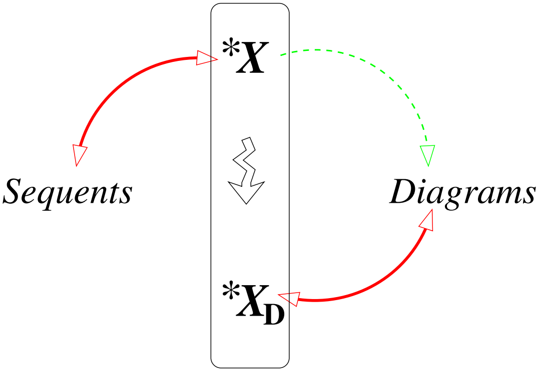
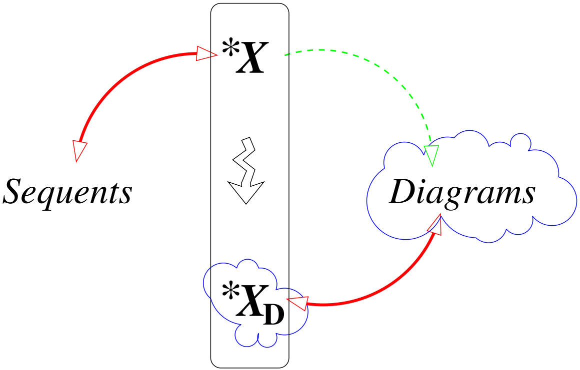
Outline



*X
Sequents
Diagrams
*X_D



*X
Sequents
Diagrams
*X_D

Introduction

- Computationl interpretation of classical proofs
- Sequent calculus with explicit structural rules (Weakening and Contraction)
- 1) Terms in $^*\mathcal{X}$-calculus are in fact annotations for proofs,
  2) Computation in $^*\mathcal{X}$-calculus corresponds to cut-elimination.
- $\mathcal{X}$-calculus is a precursor of $^*\mathcal{X}$
  Corresponding to the sequent system
  where structural rules are hidden.

Names

Terms are built from *names*.

Two categories of names : $x, y, z...$ in-names
$\alpha, \beta, \gamma...$ out-names

Bound names wear "hats" : $\widehat{x}, \widehat{y}, \widehat{z}... \ \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}...$

Examples :

$\langle x.\alpha\rangle \qquad \widehat{x}\,\langle x.\beta\rangle\,\widehat{\beta}\cdot\alpha \qquad x{<}^{\widehat{y}}_{\widehat{z}}\left[M\right] \qquad \left[N\right]^{\widehat{\beta}}_{\widehat{\gamma}}{>}\alpha \qquad P\widehat{\alpha}\ \dagger\ \widehat{x}Q$

Name $\neq$ Variable

- In λ-calculus : variables
  - $(\lambda y.xyz)M \xrightarrow{\beta} xMz$
  - An arbitrary term M substitutes the variable y
- In $*\mathcal{X}$-calculus : names
  - A name can never be substituted for a term
  - A name can only be renamed

G1 sequent system for classical logic

$$\frac{}{A \vdash A}\ (axiom)$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \to B \vdash \Delta, \Delta'}\ (\to\ left) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to\ right)$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ (cut)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ (left\ \ weakening) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ (right\ \ weakening)$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ (left\ \ contraction) \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ (right\ \ contraction)$$

Curry-Howard-de Bruijn Correspondence

- Reveals a strong conection between logic and computation
- Originaly the notion is used to relate Intuitionistic Logic in ND, with simply typed λ-calculus
- "Curry-Howard paradigm".
- This work : Classical Logic in SC (G1), with typed $*\mathcal{X}$-calculus

Proofs	$\Leftrightarrow$	*Terms*
Propositions	$\Leftrightarrow$	*Types*
Cut–eliminations	$\Leftrightarrow$	*Reductions*

- We are speaking about *computational interpretation* of classical proofs.

Terms correspond to proofs :

$$\frac{}{\langle x.\alpha\rangle \because \ x : A \vdash \alpha : A}\ (cap)$$

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$$\frac{M :\cdot\; \Gamma \vdash \alpha : A, \Delta \qquad N :\cdot\; \Gamma', x : B \vdash \Delta'}{M\widehat{\alpha}\ [y]\ \widehat{x}N :\cdot\; \Gamma, \Gamma', y : A \to B \vdash \Delta, \Delta'}\ (imp) \qquad \frac{M :\cdot\; \Gamma, x : A \vdash \alpha : B, \Delta}{\widehat{x}\ M\ \widehat{\alpha}\cdot\beta :\cdot\; \Gamma \vdash \beta : A \to B, \Delta}\ (exp)$$

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$$\frac{M :\cdot\ \Gamma \vdash \Delta}{x \odot M :\cdot\ \Gamma, x : A \vdash \Delta}\ (left\ \ eraser) \qquad \frac{M :\cdot\ \Gamma \vdash \Delta}{M \odot \alpha :\cdot\ \Gamma \vdash \alpha : A, \Delta}\ (right\ \ eraser)$$

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$$\frac{M \because\; \Gamma \vdash \Delta}{x \odot M \because\; \Gamma, x : A \vdash \Delta}\;(left\;\;eraser) \qquad \frac{M \because\; \Gamma \vdash \Delta}{M \odot \alpha \because\; \Gamma \vdash \alpha : A, \Delta}\;(right\;\;eraser)$$

$$\frac{M \because\; \Gamma, x : A, y : A \vdash \Delta}{z{<}^{\widehat{x}}_{\widehat{y}}\big[M\big] \because\; \Gamma, z : A \vdash \Delta}\;(left\;\;dupl.) \qquad \frac{M \because\; \Gamma \vdash \alpha : A, \beta : A, \Delta}{\big[M\big]^{\widehat{\alpha}}_{\widehat{\beta}}{>}\gamma \because\; \Gamma \vdash \gamma : A, \Delta}\;(right\;\;dupl.)$$

*$\mathcal{X}$-calculus

The syntax

$$
\begin{array}{lcll}
M, N & ::= & \langle x.\alpha \rangle & \textit{capsule} \\
 & \mid & \widehat{x}\, M\, \widehat{\beta} \cdot \alpha & \textit{exporter} \\
 & \mid & M \widehat{\alpha}\, [y]\, \widehat{x} N & \textit{importer} \\
 & \mid & M \widehat{\alpha} \dagger \widehat{x} N & \textit{cut} \\
 & \mid & x \odot M & \textit{left-eraser} \\
 & \mid & M \odot \alpha & \textit{right-eraser} \\
 & \mid & z <^{\widehat{x}}_{\widehat{y}} [M] & \textit{left-duplicator} \\
 & \mid & [M]^{\widehat{\alpha}}_{\widehat{\beta}} > \gamma & \textit{right-duplicator}
\end{array}
$$

- The notion of a principal name of a term.
  1. L–principal name
  2. S–principal name

Linearity

- In $^*\mathcal{X}$ we consider only the linear terms :
  - ◇ every free name occurs only once
  - ◇ every binder does bind an occurence of a free name (and therefore only one)

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$$\widehat{x}\,(x \odot \langle y.\beta\rangle)\,\widehat{\beta}\cdot\alpha$$

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$$\widehat{x}\,(x\odot\langle y.\beta\rangle)\,\widehat{\beta}\cdot\alpha \qquad \text{and} \qquad \Big[\langle x.\alpha_1\rangle \odot \alpha_2\Big]^{\widehat{\alpha_1}}_{\widehat{\alpha_2}} > \alpha$$

Formal definition :

$$\frac{}{\langle x.\alpha\rangle \ \ linear}$$

$$\frac{M \ \ linear, \ \ x \in f_{in}(M), \beta \in f_{on}(M), \ \ \alpha \notin f_{on}(M)}{\widehat{x}\, M\, \widehat{\beta} \cdot \alpha \ \ linear}$$

$$\frac{M, N \ \ linear, \ \ \alpha \in f_{on}(M), \ \ x \in f_{in}(N), \ \ y \notin f_{in}(M,N), \ \ f_n(M) \cap f_n(N) = \emptyset}{M\, \widehat{\alpha}\, [y]\, \widehat{x}\, N \ \ linear}$$

$$\frac{M, N \ \ linear, \ \ \alpha \in f_{on}(M), \ \ x \in f_{in}(N), \ \ f_n(M) \cap f_n(N) = \emptyset}{M\widehat{\alpha} \ \dagger \ \widehat{x}N \ \ linear}$$

$$\frac{M \ \ linear, \ \ x \notin f_{in}(M)}{x \odot M \ \ linear} \qquad \frac{M \ \ linear, \ \ \alpha \notin f_{on}(M)}{M \odot \alpha \ \ linear}$$

$$\frac{M \ \ linear, \ \ x, y \in f_{in}(M), \ \ z \notin f_{in}(M)}{z {<} {\widehat{x} \atop \widehat{y}} \left[M\right] \ \ linear} \qquad \frac{M \ \ linear, \ \ \alpha, \beta \in f_{on}(M), \ \ \gamma \notin f_{on}(M)}{\left[M\right] {\widehat{\alpha} \atop \widehat{\beta}} {>} \gamma \ \ linear}$$

- Linearity of terms enables us to draw them

Reduction rules are grouped into :

- Logical rules (L-principal names involved)

- Activation rules
- Deactivation rules

- Structural rules (S-principal names involved)
- Propagation rules

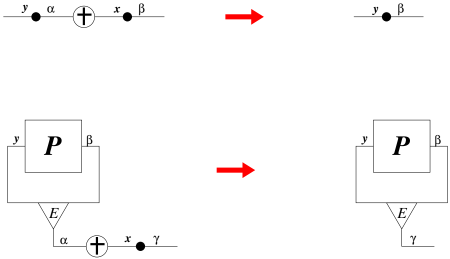
$*\mathcal{X}$-calculus

Reduction rules

LOGICAL RULES :

$$
\begin{array}{llcll}
(cap - cap) & : & \langle y.\alpha\rangle\widehat{\alpha} \ \dagger \ \widehat{x}\langle x.\beta\rangle & \rightarrow & \langle y.\beta\rangle \\
(exp - cap) & : & (\widehat{y}\ P\ \widehat{\beta}\cdot\alpha)\widehat{\alpha} \ \dagger \ \widehat{x}\langle x.\gamma\rangle & \rightarrow & \widehat{y}\ P\ \widehat{\beta}\cdot\gamma \\
(cap - imp) & : & \langle y.\alpha\rangle\widehat{\alpha} \ \dagger \ \widehat{x}(P\widehat{\beta}\ [x]\ \widehat{z}Q) & \rightarrow & P\widehat{\beta}\ [y]\ \widehat{z}Q \\
(exp - imp) & : & (\widehat{y}\ P\ \widehat{\beta}\cdot\alpha)\widehat{\alpha} \ \dagger \ \widehat{x}(Q\widehat{\gamma}\ [x]\ \widehat{z}R) & \rightarrow & either \left\{ \begin{array}{l} (Q\widehat{\gamma} \ \dagger \ \widehat{y}P)\widehat{\beta} \ \dagger \ \widehat{z}R \\ Q\widehat{\gamma} \ \dagger \ \widehat{y}(P\widehat{\beta} \ \dagger \ \widehat{z}R) \end{array} \right.
\end{array}
$$

cap-cap and exp-cap (merging)



*$\mathcal{X}$-calculus

Reduction rules

- Cuts can be left-activated or right-activated

ACTIVATION RULES :

$$
\begin{array}{llll}
(act - L) & : & P\widehat{\alpha} \dagger \widehat{x}Q \rightarrow P\widehat{\alpha} \not\nearrow \widehat{x}Q, & \text{with } P \neq \widehat{y}\, M\, \widehat{\beta} \cdot \alpha \text{ and } P \neq \langle y.\alpha \rangle \\
(act - R) & : & P\widehat{\alpha} \dagger \widehat{x}Q \rightarrow P\widehat{\alpha} \not\nwarrow \widehat{x}Q, & \text{with } Q \neq M\widehat{\beta}\, [x]\, \widehat{y}N \text{ and } Q \neq \langle x.\beta \rangle
\end{array}
$$

- Extending the syntax with two new elements

$$
\begin{array}{llll}
M, N & ::= & \cdots & \\
 & | & M\widehat{\alpha} \not\nearrow \widehat{x}N & \textit{left–cut} \\
 & | & M\widehat{\alpha} \not\nwarrow \widehat{x}N & \textit{right–cut}
\end{array}
$$

DEACTIVATION RULES

Left :

$$
\begin{array}{lcll}
(cap \not\nearrow - deactivation) & : & \langle x.\beta\rangle\widehat{\beta} \not\nearrow \widehat{y}R & \rightarrow \quad \langle x.\beta\rangle\widehat{\beta} \dagger \widehat{y}R \\
(exp \not\nearrow - deactivation) & : & (\widehat{x}\, P\, \widehat{\gamma} \cdot \beta)\widehat{\beta} \not\nearrow \widehat{y}R & \rightarrow \quad (\widehat{x}\, P\, \widehat{\gamma} \cdot \beta)\widehat{\beta} \dagger \widehat{y}R
\end{array}
$$

Right :

$$
\begin{array}{lcll}
(\not\nwarrow cap - deactivation) & : & P\widehat{\alpha} \not\nwarrow \widehat{x}\langle x.\beta\rangle & \rightarrow \quad P\widehat{\alpha} \dagger \widehat{x}\langle x.\beta\rangle \\
(\not\nwarrow imp - deactivation) & : & P\widehat{\alpha} \not\nwarrow \widehat{x}(Q\widehat{\beta}\, [x]\, \widehat{y}R) & \rightarrow \quad P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\, [x]\, \widehat{y}R)
\end{array}
$$

*$\mathcal{X}$-calculus

Reduction rules

STRUCTURAL RULES

Left :

$$(\not\nearrow \textit{erasure}) \quad : \quad (P \odot \beta)\widehat{\beta} \not\nearrow \widehat{y}Q \quad \rightarrow \quad \mathcal{I}^Q \odot P \odot \mathcal{O}^Q$$

$$(\not\nearrow \textit{duplication}) \quad : \quad ([P]\frac{\widehat{\beta_1}}{\widehat{\beta_2}} > \beta)\widehat{\beta} \not\nearrow \widehat{y}Q \quad \rightarrow \quad \mathcal{I}^Q < \frac{\widehat{\mathcal{I}_1^Q}}{\mathcal{I}_2^Q} \left[(P\widehat{\beta_1} \not\nearrow \widehat{y_1}Q_1)\widehat{\beta_2} \not\nearrow \widehat{y_2}Q_2\right] \frac{\widehat{\mathcal{O}_1^Q}}{\mathcal{O}_2^Q} > \mathcal{O}^Q$$

Where :

$\mathcal{I}^Q = \bar{f_{in}}(Q) \setminus y, \quad \mathcal{O}^Q = \bar{f_{on}}(Q)$ and

$Q_i = ind(Q, \ f_n(Q), \ i)$ for $i = 1, 2$

Right :

$$(\not\nwarrow \textit{erasure}) \quad : \quad P\widehat{\alpha} \not\nwarrow \widehat{x}(x \odot Q) \quad \longrightarrow \quad \mathcal{I}^P \odot Q \odot \mathcal{O}^P$$

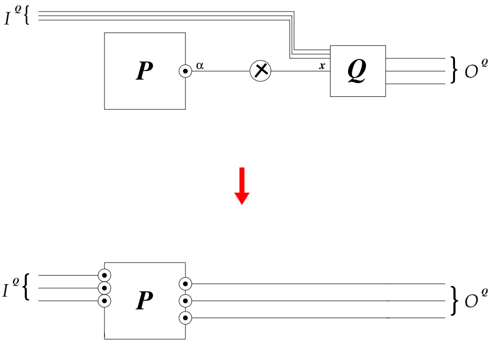
$$(\not\nwarrow \textit{duplication}) \quad : \quad P\widehat{\alpha} \not\nwarrow \widehat{x}(x < \frac{\widehat{x_1}}{\widehat{x_2}}[Q]) \quad \longrightarrow \quad \mathcal{I}^P < \frac{\widehat{\mathcal{I}_1^P}}{\mathcal{I}_2^P} \left[P_2\widehat{\alpha_2} \not\nwarrow \widehat{x_2}(P_1\widehat{\alpha_1} \not\nwarrow \widehat{x_1}Q)\right] \frac{\widehat{\mathcal{O}_1^P}}{\mathcal{O}_2^P} > \mathcal{O}^P$$

Where :

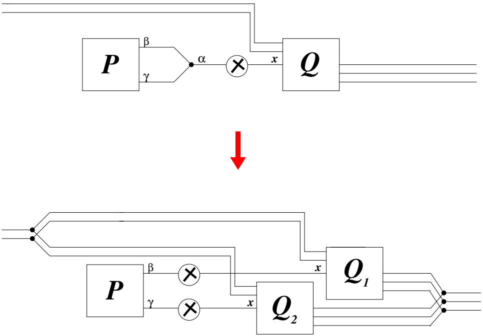
$\mathcal{I}^P = \bar{f_{in}}(P), \quad \mathcal{O}^P = \bar{f_{on}}(P) \setminus \alpha$ and

$P_i = ind(P, \ f_n(P), \ i)$ for $i = 1, 2$

⤫-erasure



⨯̸-duplication



*$\mathcal{X}$-calculus

Reduction rules

PROPAGATION RULES

Left :

$$
\begin{array}{llll}
(exp \not\nearrow - prop) & : (\widehat{x}\, P\, \widehat{\gamma} \cdot \alpha)\widehat{\beta} \not\nearrow \widehat{y}R & \rightarrow \widehat{x}\,(P\widehat{\beta} \not\nearrow \widehat{y}R)\, \widehat{\gamma} \cdot \alpha & \\
(imp \not\nearrow - prop_1) & : (P\widehat{\alpha}\ [x]\ \widehat{z}Q)\widehat{\beta} \not\nearrow \widehat{y}R & \rightarrow (P\widehat{\beta} \not\nearrow \widehat{y}R)\widehat{\alpha}\ [x]\ \widehat{z}Q, & \beta \in f_{on}(P) \\
(imp \not\nearrow - prop_2) & : (P\widehat{\alpha}\ [x]\ \widehat{z}Q)\widehat{\beta} \not\nearrow \widehat{y}R & \rightarrow P\widehat{\alpha}\ [x]\ \widehat{z}(Q\widehat{\beta} \not\nearrow \widehat{y}R), & \beta \in f_{on}(Q) \\
 & & & \\
(cut(caps) \not\nearrow - prop) & : (P\widehat{\alpha}\ \dagger\ \widehat{x}\langle x.\beta\rangle)\widehat{\beta} \not\nearrow \widehat{y}R & \rightarrow P\widehat{\alpha}\ \dagger\ \widehat{y}R & \\
(cut \not\nearrow - prop_1) & : (P\widehat{\alpha}\ \dagger\ \widehat{x}Q)\widehat{\beta} \not\nearrow \widehat{y}R & \rightarrow (P\widehat{\beta} \not\nearrow \widehat{y}R)\widehat{\alpha}\ \dagger\ \widehat{x}Q, & \beta \in f_{on}(P),\ q \neq \langle x.\beta\rangle \\
(cut \not\nearrow - prop_2) & : (P\widehat{\alpha}\ \dagger\ \widehat{x}Q)\widehat{\beta} \not\nearrow \widehat{y}R & \rightarrow P\widehat{\alpha}\ \dagger\ \widehat{x}(Q\widehat{\beta} \not\nearrow \widehat{y}R), & \beta \in f_{on}(Q),\ q \neq \langle x.\beta\rangle \\
 & & & \\
(L\text{–}eras \not\nearrow - prop) & : (x \odot M)\widehat{\beta} \not\nearrow \widehat{y}R & \rightarrow x \odot (M\widehat{\beta} \not\nearrow \widehat{y}R) & \\
(R\text{–}eras \not\nearrow - prop) & : (M \odot \alpha)\widehat{\beta} \not\nearrow \widehat{y}R & \rightarrow (M\widehat{\beta} \not\nearrow \widehat{y}R) \odot \alpha, & \alpha \neq \beta \\
 & & & \\
(L\text{–}dupl \not\nearrow - prop) & : (x {<} ^{\widehat{x_1}}_{\widehat{x_2}} \big[M\big])\widehat{\beta} \not\nearrow \widehat{y}R & \rightarrow x {<} ^{\widehat{x_1}}_{\widehat{x_2}} \big[M\widehat{\beta} \not\nearrow \widehat{y}R\big] & \\
(R\text{–}dupl \not\nearrow - prop) & : (\big[M\big] ^{\widehat{\alpha_1}}_{\widehat{\alpha_2}} {>} \alpha)\widehat{\beta} \not\nearrow \widehat{y}R & \rightarrow \big[M\widehat{\beta} \not\nearrow \widehat{y}R\big] ^{\widehat{\alpha_1}}_{\widehat{\alpha_2}} {>} \alpha, & \alpha \neq \beta
\end{array}
$$

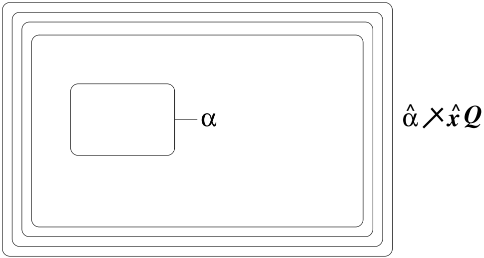
$*\mathcal{X}$-calculus

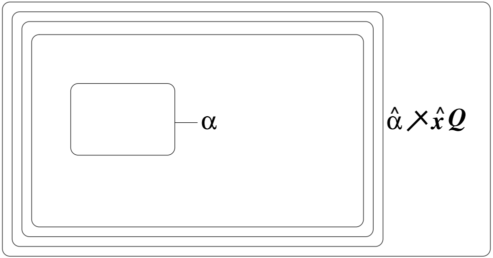
Reduction rules

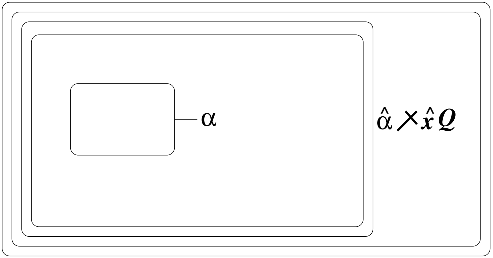
PROPAGATION RULES

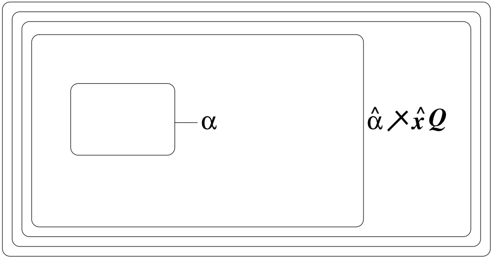
Right :

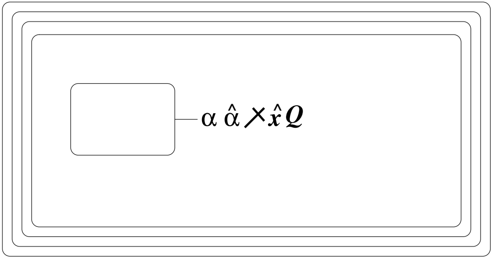
$(\not\nwarrow exp - prop)$	:	$P\widehat{\alpha} \not\nwarrow \widehat{x}(\widehat{y}\ Q\ \widehat{\beta} \cdot \gamma)$	$\rightarrow$	$\widehat{y}\ (P\widehat{\alpha} \not\nwarrow \widehat{x}Q)\ \widehat{\beta} \cdot \gamma$	
$(\not\nwarrow imp - prop_1)$	:	$P\widehat{\alpha} \not\nwarrow \widehat{x}(Q\widehat{\beta}\ [y]\ \widehat{z}R)$	$\rightarrow$	$(P\widehat{\alpha} \not\nwarrow \widehat{x}Q)\widehat{\beta}\ [y]\ \widehat{z}R,$	$x \in f_{in}(Q)$
$(\not\nwarrow imp - prop_2)$	:	$P\widehat{\alpha} \not\nwarrow \widehat{x}(Q\widehat{\beta}\ [y]\ \widehat{z}R)$	$\rightarrow$	$Q\widehat{\beta}\ [y]\ \widehat{z}(P\widehat{\alpha} \not\nwarrow \widehat{x}R),$	$x \in f_{in}(R)$
$(\not\nwarrow cut(caps) - prop)$	:	$P\widehat{\alpha} \not\nwarrow \widehat{x}(\langle x.\beta\rangle\widehat{\beta}\ \dagger\ \widehat{y}R)$	$\rightarrow$	$P\widehat{\alpha}\ \dagger\ \widehat{y}R$	
$(\not\nwarrow cut - prop_1)$	:	$P\widehat{\alpha} \not\nwarrow \widehat{x}(Q\widehat{\beta}\ \dagger\ \widehat{y}R)$	$\rightarrow$	$(P\widehat{\alpha} \not\nwarrow \widehat{x}Q)\widehat{\beta}\ \dagger\ \widehat{y}R,$	$x \in f_{in}(Q),\ Q \neq \langle x.\beta\rangle$
$(\not\nwarrow cut - prop_2)$	:	$P\widehat{\alpha} \not\nwarrow \widehat{x}(Q\widehat{\beta}\ \dagger\ \widehat{y}R)$	$\rightarrow$	$Q\widehat{\beta}\ \dagger\ \widehat{y}(P\widehat{\alpha} \not\nwarrow \widehat{x}R),$	$x \in f_{in}(R),\ Q \neq \langle x.\beta\rangle$
$(\not\nwarrow L\text{-}eras - prop)$	:	$P\widehat{\alpha} \not\nwarrow \widehat{x}(y \odot Q)$	$\rightarrow$	$y \odot (P\widehat{\alpha} \not\nwarrow \widehat{x}Q),$	$x \neq y$
$(\not\nwarrow R\text{-}eras - prop)$	:	$P\widehat{\alpha} \not\nwarrow \widehat{x}(Q \odot \beta)$	$\rightarrow$	$(P\widehat{\alpha} \not\nwarrow \widehat{x}Q) \odot \beta$	
$(\not\nwarrow L\text{-}dupl - prop)$	:	$P\widehat{\alpha} \not\nwarrow \widehat{x}(y{<}\frac{\widehat{y_1}}{\widehat{y_2}}\left[Q\right])$	$\rightarrow$	$y{<}\frac{\widehat{y_1}}{\widehat{y_2}}\left[P\widehat{\alpha} \not\nwarrow \widehat{x}Q\right],$	$x \neq y$
$(\not\nwarrow R\text{-}dupl - prop)$	:	$P\widehat{\alpha} \not\nwarrow \widehat{x}(\left[Q\right]\frac{\widehat{\beta_1}}{\widehat{\beta_2}}{>}\beta)$	$\rightarrow$	$\left[P\widehat{\alpha} \not\nwarrow \widehat{x}Q\right]\frac{\widehat{\beta_1}}{\widehat{\beta_2}}{>}\beta$	

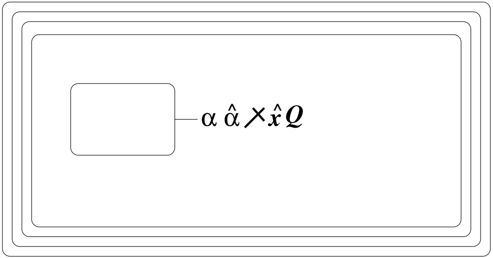

α
α̂ ⤫ x̂Q


α
α̂ ⨯ x̂ Q


α
α̂ ⨯ x̂ Q


α
α̂ ⤫ x̂ Q


$\alpha\ \hat{\alpha}\ \times\ \hat{x}\ Q$



- $^*\mathcal{X}$ is basically the calculus of explicit substitutions

Properties

Reduction relation in $^*\mathcal{X}$:

1. Preserves linearity of terms

   If P is linear and $P \xrightarrow{*\mathcal{X}} Q$ then Q is linear

2. Preserves the set of free names

   If $P \xrightarrow{*\mathcal{X}} Q$ then $f_n(P) = f_n(Q)$

   - Lafont's interface preservation

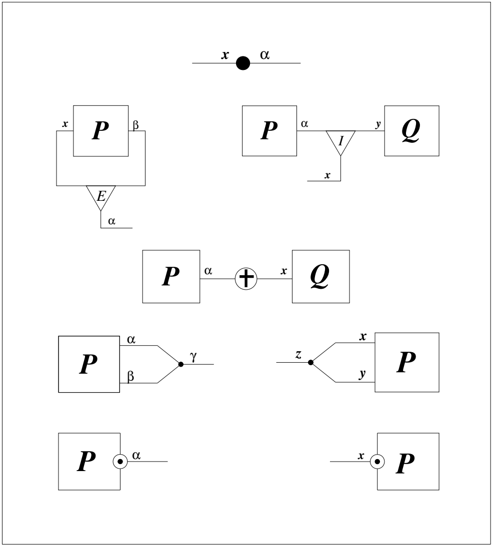
3. Subject reduction

   $\textit{If} \;\; S \;:\cdot\; \Gamma \vdash \Delta \;\; \textit{and} \;\; S \xrightarrow{*\mathcal{X}} S', \;\; \textit{then} \;\; S' \;:\cdot\; \Gamma \vdash \Delta$

4. Strong normalisation ?

Diagrammatic calculus

The syntax



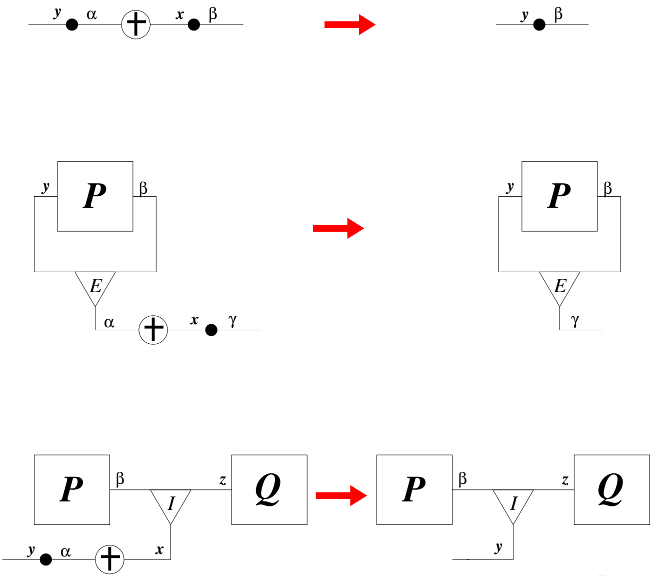
Diagrammatic calculus

Reduction rules

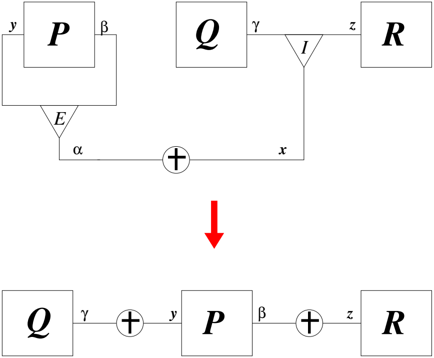
- Logical
- Activation
- Structural
- Deactivation

- no propagation rules

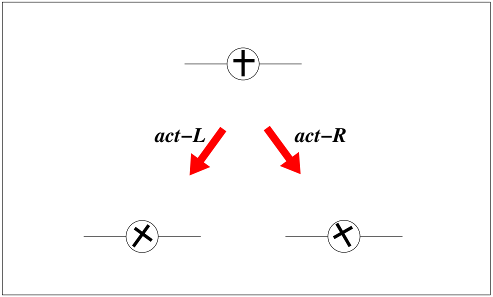
Logical actions (merging)



Logical actions (inserting)

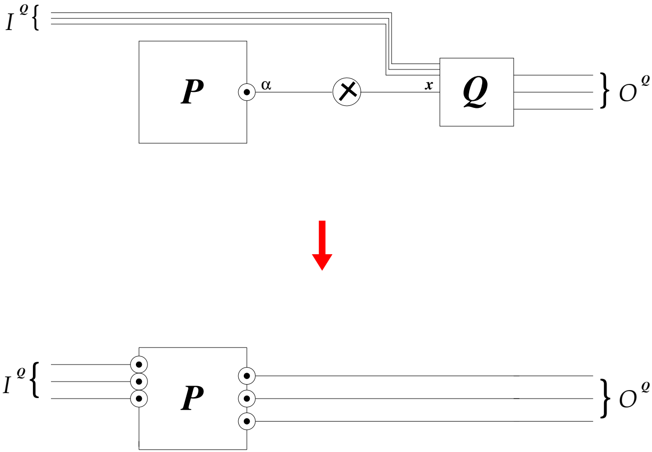


Activation rules



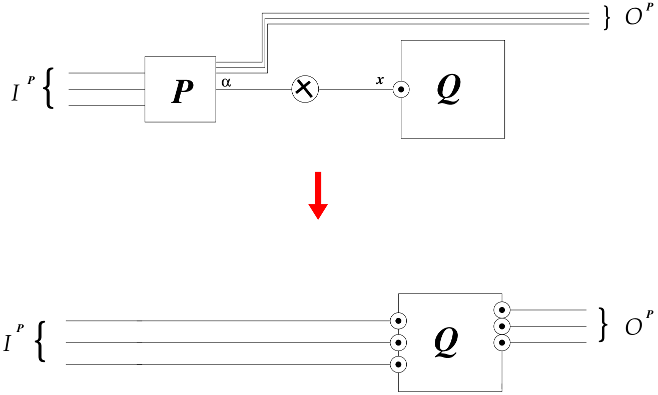
Structural rules

⅋-Erasure



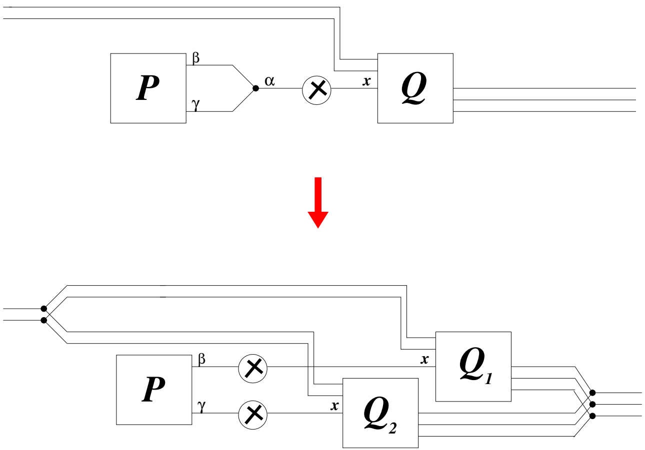
Structural rules

⅄-Erasure



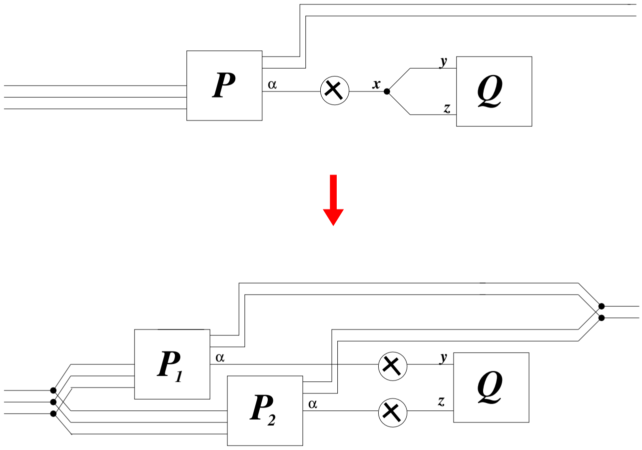
Structural rules

⅋-Duplication

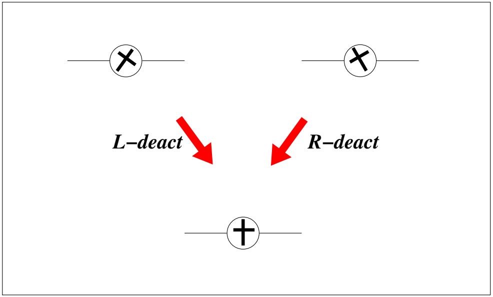


Structural rules

⅋-Duplication



Deactivations



- In total 14 reduction rules

$^*\mathcal{X}_D$-calculus

The idea

- Modify $^*\mathcal{X}$ to get a correspondence with Diagrammatic calculus
- The new variant is called $^*\mathcal{X}_D$
- It introduces congruence relation $(\equiv)$ on terms
- Reducing is - reducing modulo the congruence relation

$*\mathcal{X}_D$-calculus

The syntax

M, N	$::=$	$\langle x.\alpha \rangle$	*capsule*
	$\mid$	$\widehat{x}\, M\, \widehat{\beta} \cdot \alpha$	*exporter*
	$\mid$	$M \widehat{\alpha}\ [y]\ \widehat{x} N$	*importer*
	$\mid$	$M \widehat{\alpha}\ \dagger\ \widehat{x} N$	*cut*
	$\mid$	$x \odot M$	*left-eraser*
	$\mid$	$M \odot \alpha$	*right-eraser*
	$\mid$	$z{<}^{\widehat{x}}_{\widehat{y}}[M]$	*left-duplicator*
	$\mid$	$[M]^{\widehat{\alpha}}_{\widehat{\beta}}{>}\gamma$	*right-duplicator*

LOGICAL RULES :

$(cap - cap)$	:	$\langle y.\alpha\rangle\widehat{\alpha} \dagger \widehat{x}\langle x.\beta\rangle$	$\rightarrow$	$\langle y.\beta\rangle$
$(exp - cap)$	:	$(\widehat{y}\ P\ \widehat{\beta}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}\langle x.\gamma\rangle$	$\rightarrow$	$\widehat{y}\ P\ \widehat{\beta}\cdot\gamma$
$(cap - imp)$	:	$\langle y.\alpha\rangle\widehat{\alpha} \dagger \widehat{x}(P\widehat{\beta}\ [x]\ \widehat{z}Q)$	$\rightarrow$	$P\widehat{\beta}\ [y]\ \widehat{z}Q$
$(exp - imp)$	:	$(\widehat{y}\ P\ \widehat{\beta}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\gamma}\ [x]\ \widehat{z}R)$	$\rightarrow$	$Q\widehat{\gamma} \dagger \widehat{y}(P\widehat{\beta} \dagger \widehat{z}R)$

$*\mathcal{X}_D$-calculus

Reduction rules

ACTIVATION RULES :

$$
\begin{array}{llll}
(act - L) & : & P\widehat{\alpha} \dagger \widehat{x}Q \rightarrow P\widehat{\alpha} \nearrow \widehat{x}Q, & \text{if } \alpha \text{ not L-principal for } P \\
 & & & \text{nor for any } M \in subt\{P\} \\
(act - R) & : & P\widehat{\alpha} \dagger \widehat{x}Q \rightarrow P\widehat{\alpha} \nwarrow \widehat{x}Q, & \text{if } x \text{ not L-principal for } Q \\
 & & & \text{nor for any } N \in subt\{Q\}
\end{array}
$$

DEACTIVATION RULES

Left :

$$
\begin{array}{llll}
(cap\nearrow - deactivation) & : & \langle x.\beta\rangle\widehat{\beta} \nearrow \widehat{y}R & \rightarrow \langle x.\beta\rangle\widehat{\beta} \dagger \widehat{y}R \\
(exp\nearrow - deactivation) & : & (\widehat{x}\, P\, \widehat{\gamma} \cdot \beta)\widehat{\beta} \nearrow \widehat{y}R & \rightarrow (\widehat{x}\, P\, \widehat{\gamma} \cdot \beta)\widehat{\beta} \dagger \widehat{y}R
\end{array}
$$

Right :

$$
\begin{array}{llll}
(\nwarrow cap - deactivation) & : & P\widehat{\alpha} \nwarrow \widehat{x}\langle x.\beta\rangle & \rightarrow P\widehat{\alpha} \dagger \widehat{x}\langle x.\beta\rangle \\
(\nwarrow imp - deactivation) & : & P\widehat{\alpha} \nwarrow \widehat{x}(Q\widehat{\beta}\,[x]\,\widehat{y}R) & \rightarrow P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\,[x]\,\widehat{y}R)
\end{array}
$$

PROPAGATION RULES

Left :

$$
\begin{array}{llll}
(exp\not\nearrow - prop) & : (\widehat{x}\,P\,\widehat{\gamma}\cdot\alpha)\widehat{\beta}\not\nearrow\widehat{y}R & \to \widehat{x}\,(P\widehat{\beta}\not\nearrow\widehat{y}R)\,\widehat{\gamma}\cdot\alpha & \\
(imp\not\nearrow - prop_1) & : (P\widehat{\alpha}\,[x]\,\widehat{z}Q)\widehat{\beta}\not\nearrow\widehat{y}R & \to (P\widehat{\beta}\not\nearrow\widehat{y}R)\widehat{\alpha}\,[x]\,\widehat{z}Q, & \beta\in f_{on}(P) \\
(imp\not\nearrow - prop_2) & : (P\widehat{\alpha}\,[x]\,\widehat{z}Q)\widehat{\beta}\not\nearrow\widehat{y}R & \to P\widehat{\alpha}\,[x]\,\widehat{z}(Q\widehat{\beta}\not\nearrow\widehat{y}R), & \beta\in f_{on}(Q) \\
 & & & \\
(cut(caps)\not\nearrow - prop) & : (P\widehat{\alpha}\dagger\widehat{x}\langle x.\beta\rangle)\widehat{\beta}\not\nearrow\widehat{y}R & \to P\widehat{\alpha}\dagger\widehat{y}R & \\
(cut\not\nearrow - prop_1) & : (P\widehat{\alpha}\dagger\widehat{x}Q)\widehat{\beta}\not\nearrow\widehat{y}R & \to (P\widehat{\beta}\not\nearrow\widehat{y}R)\widehat{\alpha}\dagger\widehat{x}Q, & \beta\in f_{on}(P),\ q\neq\langle x.\beta\rangle \\
(cut\not\nearrow - prop_2) & : (P\widehat{\alpha}\dagger\widehat{x}Q)\widehat{\beta}\not\nearrow\widehat{y}R & \to P\widehat{\alpha}\dagger\widehat{x}(Q\widehat{\beta}\not\nearrow\widehat{y}R), & \beta\in f_{on}(Q),\ q\neq\langle x.\beta\rangle \\
 & & & \\
(L\text{-}eras\not\nearrow - prop) & : (x\odot M)\widehat{\beta}\not\nearrow\widehat{y}R & \to x\odot(M\widehat{\beta}\not\nearrow\widehat{y}R) & \\
(R\text{-}eras\not\nearrow - prop) & : (M\odot\alpha)\widehat{\beta}\not\nearrow\widehat{y}R & \to (M\widehat{\beta}\not\nearrow\widehat{y}R)\odot\alpha, & \alpha\neq\beta \\
 & & & \\
(L\text{-}dupl\not\nearrow - prop) & : (x{<}^{\widehat{x_1}}_{\widehat{x_2}}[M])\widehat{\beta}\not\nearrow\widehat{y}R & \to x{<}^{\widehat{x_1}}_{\widehat{x_2}}[M\widehat{\beta}\not\nearrow\widehat{y}R] & \\
(R\text{-}dupl\not\nearrow - prop) & : ([M]^{\widehat{\alpha_1}}_{\widehat{\alpha_2}}{>}\alpha)\widehat{\beta}\not\nearrow\widehat{y}R & \to [M\widehat{\beta}\not\nearrow\widehat{y}R]^{\widehat{\alpha_1}}_{\widehat{\alpha_2}}{>}\alpha, & \alpha\neq\beta
\end{array}
$$

No propagation rules !

PROPAGATION RULES

Right :

$(\nwarrow exp - prop)$	:	$P\widehat{\alpha} \nwarrow \widehat{x}(\widehat{y}\ Q\ \widehat{\beta} \cdot \gamma)$	$\rightarrow$	$\widehat{y}\ (P\widehat{\alpha} \nwarrow \widehat{x}Q)\ \widehat{\beta} \cdot \gamma$	
$(\nwarrow imp - prop_1)$	:	$P\widehat{\alpha} \nwarrow \widehat{x}(Q\widehat{\beta}\ [y]\ \widehat{z}R)$	$\rightarrow$	$(P\widehat{\alpha} \nwarrow \widehat{x}Q)\widehat{\beta}\ [y]\ \widehat{z}R,$	$x \in f_{in}(Q)$
$(\nwarrow imp - prop_2)$	:	$P\widehat{\alpha} \nwarrow \widehat{x}(Q\widehat{\beta}\ [y]\ \widehat{z}R)$	$\rightarrow$	$Q\widehat{\beta}\ [y]\ \widehat{z}(P\widehat{\alpha} \nwarrow \widehat{x}R),$	$x \in f_{in}(R)$
$(\nwarrow cut(caps) - prop)$	:	$P\widehat{\alpha} \nwarrow \widehat{x}(\langle x.\beta\rangle\widehat{\beta} \dagger \widehat{y}R)$	$\rightarrow$	$P\widehat{\alpha} \dagger \widehat{y}R$	
$(\nwarrow cut - prop_1)$	:	$P\widehat{\alpha} \nwarrow \widehat{x}(Q\widehat{\beta} \dagger \widehat{y}R)$	$\rightarrow$	$(P\widehat{\alpha} \nwarrow \widehat{x}Q)\widehat{\beta} \dagger \widehat{y}R,$	$x \in f_{in}(Q),\ Q \neq \langle x.\beta\rangle$
$(\nwarrow cut - prop_2)$	:	$P\widehat{\alpha} \nwarrow \widehat{x}(Q\widehat{\beta} \dagger \widehat{y}R)$	$\rightarrow$	$Q\widehat{\beta} \dagger \widehat{y}(P\widehat{\alpha} \nwarrow \widehat{x}R),$	$x \in f_{in}(R),\ Q \neq \langle x.\beta\rangle$
$(\nwarrow L\text{-}eras - prop)$	:	$P\widehat{\alpha} \nwarrow \widehat{x}(y \odot Q)$	$\rightarrow$	$y \odot (P\widehat{\alpha} \nwarrow \widehat{x}Q),$	$x \neq y$
$(\nwarrow R\text{-}eras - prop)$	:	$P\widehat{\alpha} \nwarrow \widehat{x}(Q \odot \beta)$	$\rightarrow$	$(P\widehat{\alpha} \nwarrow \widehat{x}Q) \odot \beta$	
$(\nwarrow L\text{-}dupl - prop)$	:	$P\widehat{\alpha} \nwarrow \widehat{x}(y{<}^{\widehat{y_1}}_{\widehat{y_2}}\left[Q\right])$	$\rightarrow$	$y{<}^{\widehat{y_1}}_{\widehat{y_2}}\left[P\widehat{\alpha} \nwarrow \widehat{x}Q\right],$	$x \neq y$
$(\nwarrow R\text{-}dupl - prop)$	:	$P\widehat{\alpha} \nwarrow \widehat{x}(\left[Q\right]^{\widehat{\beta_1}}_{\widehat{\beta_2}}{>}\beta)$	$\rightarrow$	$\left[P\widehat{\alpha} \nwarrow \widehat{x}Q\right]^{\widehat{\beta_1}}_{\widehat{\beta_2}}{>}\beta$	

No propagation rules !

$*\mathcal{X}_D$-calculus

Reduction rules

STRUCTURAL RULES

Left :

$$(\nearrow \textit{erasure}) \quad : \quad (P \odot \beta)\widehat{\beta} \nearrow \widehat{y}Q \quad \rightarrow \quad \mathcal{I}^Q \odot P \odot \mathcal{O}^Q$$

$$(\nearrow \textit{duplication}) \quad : \quad (\left[P\right]\frac{\widehat{\beta_1}}{\widehat{\beta_2}}{>}\beta)\widehat{\beta} \nearrow \widehat{y}Q \quad \rightarrow \quad \mathcal{I}^Q{<}\frac{\widehat{\mathcal{I}_1^Q}}{\widehat{\mathcal{I}_2^Q}}\left[(P\widehat{\beta_1} \nearrow \widehat{y_1}Q_1)\widehat{\beta_2} \nearrow \widehat{y_2}Q_2\right]\frac{\widehat{\mathcal{O}_1^Q}}{\widehat{\mathcal{O}_2^Q}}{>}\mathcal{O}^Q$$

Where :

$\mathcal{I}^Q = \bar{f_{in}}(Q) \setminus y, \quad \mathcal{O}^Q = \bar{f_{on}}(Q)$ and

$Q_i = ind(Q,\ f_n(Q),\ i)$ for $i = 1, 2$

Right :

$$(\nwarrow \textit{erasure}) \quad : \quad P\widehat{\alpha} \nwarrow \widehat{x}(x \odot Q) \quad \longrightarrow \quad \mathcal{I}^P \odot Q \odot \mathcal{O}^P$$

$$(\nwarrow \textit{duplication}) \quad : \quad P\widehat{\alpha} \nwarrow \widehat{x}(x{<}\frac{\widehat{x_1}}{\widehat{x_2}}\left[Q\right]) \quad \longrightarrow \quad \mathcal{I}^P{<}\frac{\widehat{\mathcal{I}_1^P}}{\widehat{\mathcal{I}_2^P}}\left[P_2\widehat{\alpha_2} \nwarrow \widehat{x_2}(P_1\widehat{\alpha_1} \nwarrow \widehat{x_1}Q)\right]\frac{\widehat{\mathcal{O}_1^P}}{\widehat{\mathcal{O}_2^P}}{>}\mathcal{O}^P$$

Where :

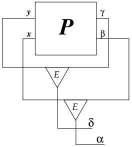
$\mathcal{I}^P = \bar{f_{in}}(P), \quad \mathcal{O}^P = \bar{f_{on}}(P) \setminus \alpha$ and

$P_i = ind(P,\ f_n(P),\ i)$ for $i = 1, 2$

We are left with 14 reduction rules, instead 34 of $*\mathcal{X}$

Congruence relation for $*\mathcal{X}_D$-terms

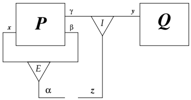
1. exp-exp

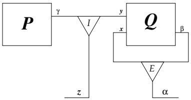


$$\widehat{y}\,(\widehat{x}\,P\,\widehat{\beta}\cdot\alpha)\,\widehat{\gamma}\cdot\delta \;\equiv\; \widehat{x}\,(\widehat{y}\,P\,\widehat{\gamma}\cdot\delta)\,\widehat{\beta}\cdot\alpha$$

with $\alpha\neq\gamma,\;\;\beta\neq\delta$

2. exp-imp





$$\widehat{x}\,(P\,\widehat{\gamma}\;[z]\;\widehat{y}\,Q)\,\widehat{\beta}\cdot\alpha \;\equiv\; (\widehat{x}\,P\,\widehat{\beta}\cdot\alpha)\,\widehat{\gamma}\;[z]\;\widehat{y}\,Q$$

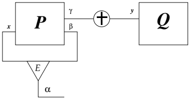
with $x,\beta\in P,\;\; x\neq z$

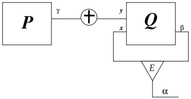
$$\widehat{x}\,(P\,\widehat{\gamma}\;[z]\;\widehat{y}\,Q)\,\widehat{\beta}\cdot\alpha \;\equiv\; P\,\widehat{\gamma}\;[z]\;\widehat{y}\,(\widehat{x}\,Q\,\widehat{\beta}\cdot\alpha)$$

with $x,\beta\in Q,\;\; x\neq z$

Congruence relation for $*\mathcal{X}_D$-terms

3. exp-cut

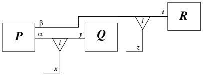


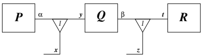


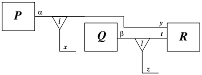
$$\widehat{x}\,(P\widehat{\gamma} \dagger \widehat{y}Q)\,\widehat{\beta}\cdot\alpha \equiv (\widehat{x}\,P\,\widehat{\beta}\cdot\alpha)\widehat{\gamma} \dagger \widehat{y}Q \qquad \text{with } x,\beta\in P$$

$$\widehat{x}\,(P\widehat{\gamma} \dagger \widehat{y}Q)\,\widehat{\beta}\cdot\alpha \equiv P\widehat{\gamma} \dagger \widehat{y}(\widehat{x}\,Q\,\widehat{\beta}\cdot\alpha) \qquad \text{with } x,\beta\in Q$$

4. imp-imp







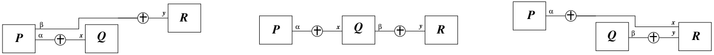
$$(P\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q)\,\widehat{\beta}\,[z]\,\widehat{t}\,R \equiv (P\,\widehat{\beta}\,[z]\,\widehat{t}\,R)\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q \qquad \text{with } \alpha,\beta\in P$$

$$(P\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q)\,\widehat{\beta}\,[z]\,\widehat{t}\,R \equiv P\,\widehat{\alpha}\,[x]\,\widehat{y}\,(Q\,\widehat{\beta}\,[z]\,\widehat{t}\,R) \qquad \text{with } y,\beta\in Q$$

$$P\,\widehat{\alpha}\,[x]\,\widehat{y}\,(Q\,\widehat{\beta}\,[z]\,\widehat{t}\,R) \equiv Q\,\widehat{\beta}\,[z]\,\widehat{t}\,(P\,\widehat{\alpha}\,[x]\,\widehat{y}\,R) \qquad \text{with } y,t\in R$$

Congruence relation for $^{*}\mathcal{X}_D$-terms

5. cut-cut



$$(P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta} \dagger \widehat{y}R \equiv (P\widehat{\beta} \dagger \widehat{y}R)\widehat{\alpha} \dagger \widehat{x}Q \qquad \text{with } \alpha, \beta \in P$$

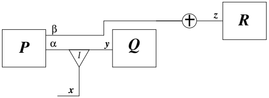
$$(P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta} \dagger \widehat{y}R \equiv P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta} \dagger \widehat{y}R) \qquad \text{with } x, \beta \in Q$$

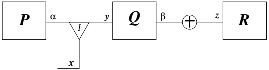
$$P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta} \dagger \widehat{y}R) \equiv Q\widehat{\beta} \dagger \widehat{y}(P\,\widehat{\alpha}\,[x]\,\widehat{R}\,) \qquad \text{with } x, y \in R$$

- ...untill now we have collected 11...
- Take a look at weakening and contraction

Congruence relation for $*\mathcal{X}_D$-terms

6a. imp-cut

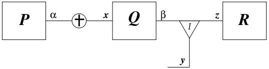


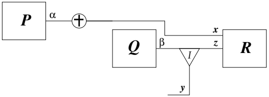


$$(P\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q)\widehat{\beta}\dagger\widehat{z}R \;\equiv\; (P\widehat{\beta}\dagger\widehat{z}R)\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q \qquad \text{with } \alpha,\beta\in P$$

$$(P\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q)\widehat{\beta}\dagger\widehat{z}R \;\equiv\; P\,\widehat{\alpha}\,[x]\,\widehat{y}\,(Q\widehat{\beta}\dagger\widehat{z}R) \qquad \text{with } y,\beta\in Q$$

6b. cut-imp



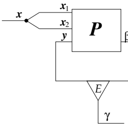


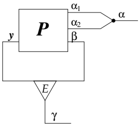
$$P\widehat{\alpha}\dagger\widehat{x}(Q\,\widehat{\beta}\,[y]\,\widehat{z}\,R) \;\equiv\; (P\widehat{\alpha}\dagger\widehat{x}Q)\,\widehat{\beta}\,[y]\,\widehat{z}\,R \qquad \text{with } x,\beta\in Q$$

$$P\widehat{\alpha}\dagger\widehat{x}(Q\,\widehat{\beta}\,[y]\,\widehat{z}\,R) \;\equiv\; Q\,\widehat{\beta}\,[y]\,\widehat{z}\,(P\widehat{\alpha}\dagger\widehat{x}R) \qquad \text{with } x,z\in R$$

Congruence relation for $*\mathcal{X}_D$-terms

7. exp-cont



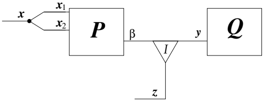


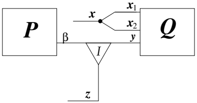
$$\widehat{x}\,(P\widehat{\gamma} \dagger \widehat{y}Q)\,\widehat{\beta}\cdot\alpha \;\equiv\; (\widehat{x}\,P\,\widehat{\beta}\cdot\alpha)\widehat{\gamma} \dagger \widehat{y}Q \qquad \text{with } x,\beta \in P$$

$$\widehat{x}\,(P\widehat{\gamma} \dagger \widehat{y}Q)\,\widehat{\beta}\cdot\alpha \;\equiv\; P\widehat{\gamma} \dagger \widehat{y}(\widehat{x}\,Q\,\widehat{\beta}\cdot\alpha) \qquad \text{with } x,\beta \in Q$$

Congruence relation for $^*\mathcal{X}_D$-terms

8. imp-contL



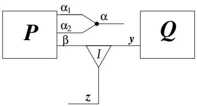


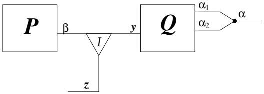
$$x\!<^{\widehat{x_1}}_{\widehat{x_2}}\!\left[P\,\widehat{\beta}\,[z]\,\widehat{y}\,Q\right] \;\equiv\; \left(x\!<^{\widehat{x_1}}_{\widehat{x_2}}\!\left[P\right]\right)\widehat{\beta}\,[z]\,\widehat{y}\,Q \qquad \text{with } x_1, x_2 \in P$$

$$x\!<^{\widehat{x_1}}_{\widehat{x_2}}\!\left[P\,\widehat{\beta}\,[z]\,\widehat{y}\,Q\right] \;\equiv\; P\,\widehat{\beta}\,[z]\,\widehat{y}\left(x\!<^{\widehat{x_1}}_{\widehat{x_2}}\!\left[Q\right]\right) \qquad \text{with } x_1, x_2 \in Q,\;\; y \neq x$$

Congruence relation for $*\mathcal{X}_D$-terms

8. imp-contR



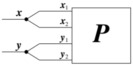


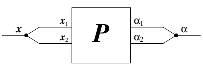
$$\left[P\,\widehat{\beta}\,[z]\,\widehat{y}\,Q\right]\widehat{\alpha_1}\widehat{\alpha_2}{>}\alpha \;\equiv\; \left(\left[P\right]\widehat{\alpha_1}\widehat{\alpha_2}{>}\alpha\right)\widehat{\beta}\,[z]\,\widehat{y}\,Q \qquad \text{with } \alpha_1,\alpha_2 \in P,\;\; \alpha \neq \beta$$

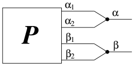
$$\left[P\,\widehat{\beta}\,[z]\,\widehat{y}\,Q\right]\widehat{\alpha_1}\widehat{\alpha_2}{>}\alpha \;\equiv\; P\,\widehat{\beta}\,[z]\,\widehat{y}\left(\left[Q\right]\widehat{\alpha_1}\widehat{\alpha_2}{>}\alpha\right) \qquad \text{with } \alpha_1,\alpha_2 \in Q$$

Congruence relation for $*\mathcal{X}_D$-terms

OO cont-cont







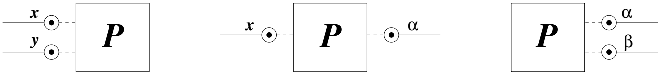
$$y < \substack{\widehat{y_1}\\ \widehat{y_2}} \left[x < \substack{\widehat{x_1}\\ \widehat{x_2}} \left[P \right] \right] \;\equiv\; x < \substack{\widehat{x_1}\\ \widehat{x_2}} \left[y < \substack{\widehat{y_1}\\ \widehat{y_2}} \left[P \right] \right] \qquad \text{with } x_1, x_2 \neq y,\;\; y_1, y_2 \neq x$$

$$\left[x < \substack{\widehat{x_1}\\ \widehat{x_2}} \left[P \right] \right] \substack{\widehat{\alpha_1}\\ \widehat{\alpha_2}} > \alpha \;\equiv\; x < \substack{\widehat{x_1}\\ \widehat{x_2}} \left[\left[P \right] \substack{\widehat{\alpha_1}\\ \widehat{\alpha_2}} > \alpha \right]$$

$$\left[\left[P \right] \substack{\widehat{\alpha_1}\\ \widehat{\alpha_2}} > \alpha \right] \substack{\widehat{\beta_1}\\ \widehat{\beta_2}} > \beta \;\equiv\; \alpha < \substack{\widehat{\alpha_1}\\ \widehat{\alpha_2}} \left[\beta < \substack{\widehat{\beta_1}\\ \widehat{\beta_2}} \left[P \right] \right] \qquad \text{with } \alpha_1, \alpha_2 \neq \beta,\;\; \beta_1, \beta_2 \neq \alpha$$

Congruence relation for $*\mathcal{X}_D$-terms

XX. weak-weak



$$
\begin{aligned}
y \odot (x \odot P) &\equiv x \odot (y \odot P) \\
x \odot (P \odot \alpha) &\equiv (x \odot P) \odot \alpha \\
(P \odot \alpha) \odot \beta &\equiv (P \odot \beta) \odot \alpha
\end{aligned}
$$

XX. cont-simplest

$$x {<}^{\widehat{x_1}}_{\widehat{x_2}} [P] \equiv x {<}^{\widehat{x_2}}_{\widehat{x_1}} [P]$$

... there are others ...

Conclusion

Other aspects :

- Link with $\mathcal{X}$.

Conclusion

Other aspects :

- Link with $\mathcal{X}$.
- Advanced constructs.

Conclusion

Other aspects :

- Link with $\mathcal{X}$.
- Advanced constructs.
- Strong normalization.

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