Towards the Fundamental Theorem of Calculus for the Lebesgue Integral in Coq

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Background

Dependencies of MathComp-Analysis

MathComp-Analysis

coq-mathcomp-classical

classical axioms

MathComp [GAA+13]

CoQ

SSReflect [Gon08]
Motivation
(besides advertising MathComp-Analysis...)

The MathComp-Analysis library:

- started with asymptotic reasoning
  - this led to a theory of derivatives [ACR18]
- extended with Lebesgue integration [AC23]
- sample applications:
  - formalization of quantum programs [ZBS+23]
  - formalization of probabilistic programs [ACS23, SA23]

⇒ Link derivatives and Lebesgue integration

⇒ Fundamental Theorem of Calculus for Lebesgue integration
The first FTC for Lebesgue integration

Statement:

- For an integrable function $f$, define $F(x) \triangleq \int_{-\infty}^{x} f(t) \, dt$.

Then $F$ is derivable and $F'(x) \overset{\text{a.e.}}{=} f(x)$.

Proofs:

- Using theorems already in MATHCOMP-ANALYSIS (the dominated convergence theorem, Fatou’s lemma, etc., see [AC23])

- ✓ As a consequence of the Lebesgue Differentiation theorem
  - whose proof requires formalization of new standard lemmas
  - which has other applications in itself
Lebesgue Differentiation theorem

Statement

Average of \( f \) over \( A \):
\[
[f]_A \triangleq \frac{1}{\mu(A)} \int_{y \in A} |f(y)|(d\mu)
\]

Definition \( \text{iavg } f A := (\text{fine (mu } A))^{-1} \cdot \int_{\mu}(y \in A) \cdot |f(y)|. \)

Deviation of \( f \) over \( B(x, r) \):
\[
f_{B(x,r)} \triangleq [\lambda y. f(y) - f(x)]_{B(x,r)}
\]

Definition \( \text{favg } f x r := \text{iavg (center (f x) \o f) (ball x r)}. \)

Lebesgue point of \( f \) at \( x \):
\[
f_{B(x,r)} \rightarrow 0 \quad r \rightarrow 0^+
\]

Definition \( \text{lebesgue_pt } f x := \text{favg } f x r @[r \rightarrow 0^'+] \rightarrow 0. \)

Lebesgue differentiation thm:

when \( f \) is locally-integrable, we have Lebesgue points a.e.

Lemma \( \text{lebesgue_differentiation } f : \) locally_integrable setT \( f \rightarrow \{\text{ae mu, forall } x, \text{lebesgue_pt } f x\}. \)
Lebesgue Differentiation theorem

Problem reduction

Lemma lebesgue_differentiation f :
  locally_integrable setT f ->
  {ae mu, forall x, lebesgue_pt f x}.

↓

Reduce the problem to

$f_k \triangleq f \mathbb{1}_{B_k}$ with $B_k \triangleq B(0, 2(k + 1))$ [Sch97, (5.12.101)]

↓

Lemma lebesgue_differentiation_bounded f :
  let B k := ball 0 k.+1.*2%:R in
  let f_ k := f \* \mathbb{1}_-(B k) in
  (forall k, mu.-integrable setT (EFin \o f_ k)) ->
  forall k, {ae mu, forall x, x \in B k -> lebesgue_pt (f_ k) x}. 
Lebesgue Differentiation theorem

**Lemma** \( \text{lebesgue\_differentiation\_bounded} \ (f : \mathbb{R} \to \mathbb{R}) : \)

\[
\text{let } B_k := \text{ball} \ 0 \ k. +1.*2%:\mathbb{R} \text{ in} \\
\text{let } f'_k := f \ \land \ \land_1(B \ k) \text{ in} \\
(\forall k, \mu.-\text{integrable set} \ (\text{EFin} \ \lor \ f'_k)) \to \\
(\forall k, \{\text{ae } \mu, \forall x, \ x \ \land B \ k \to \text{lebesgue\_pt } (f'_k) \ x\}.
\]

**Proof idea:**

- Show that \( \forall a > 0, B_k \cap \left\{ x \mid a < \limsup_{r \to 0} \frac{f_k B(x, r)}{r} \right\} \) is negligible

**...**

- ...by exhibiting continuous functions \( g_i \) such that

\[
\text{** } \subseteq \bigcap_n B_k \cap \left( \left\{ x \mid f_k(x) - g_n(x) \geq a/2 \right\} \cup \left\{ x \mid \text{HL}(f_k(x) - g_n(x)) > a/2 \right\} \right)
\]

\[
\begin{align*}
(a) & \to \text{Markov's inequality + "continuous functions dense in } L^1" \\
(b) & \to \text{Hardy-Littlewood max. ineq. } (\text{HL}(f))(x) \triangleq \sup_{r>0}\{[f]_{B(x,r)}\}
\end{align*}
\]
Lebesgue Differentiation theorem: proof organization

- Outer regularity
- Inner regularity (bounded)
  - Vitali’s lemma
  - Hardy-Littlewood maximal inequality
- Inner regularity
  - Egorov’s thm
  - Lusin’s thm
- Continuous functions are dense in $L^1$
  - Urysohn’s lemma
  - Tietze’s thm
- Lebesgue Differentiation theorem (bounded)
- FTC for Lebesgue integration
- Lebesgue’s density theorem
Sample lemma: Vitali’s covering lemma (finite case)

Context \{I : eqType\}.
Variable B : I \to set R.
Hypothesis is_ballB : forall i, is_ball (B i).
Hypothesis B_set0 : forall i, B i \neq set0.

Lemma vitali_lemma_finite s :
\{ D | \[/
{subset D <= s},
trivIset [set` D] B &
forall i, i \in s \to exists j,
[/\ j \in D,
B i \& B j \neq set0,
radius (B j) >= radius (B i) &
B i <= 3 * B j] \}.

Formalization notes:

▶ When is_ball A, the set A has a center-point and a radius.
Since A is set, a closed ball can be written closure (B i).

▶ Generalizations in MathComp-Analysis:
the infinite case of Vitali’s lemma and Vitali’s theorem
Sample lemma: Tietze’s extension theorem

Given a normal space $X$ and a closed set $A$, a function $f$ continuous on $A$ can be extended to a function $g$ continuous on the whole set while preserving boundedness.

Context $\{X : \text{topologicalType}\}$ $\{R : \text{realType}\}$ $(A : \text{set X})$.
Hypothesis normalX : normal_space X.
Hypothesis clA : closed A.

Lemma continuous_bounded_extension $f M : 0 < M \rightarrow \{\text{within } A, \text{continuous } f\} \rightarrow$
  $(\forall x, A x \rightarrow |f x| \leq M) \rightarrow$
  exists $g$, $[\forall x \in A, f = g]$, $\forall x, |g x| \leq M].$

- $\{\text{within } A, \text{continuous } f\}$ states the continuity of $f$ with a subspace topology
  - we can write $f (x + \text{eps})$, still continuity only depends on the values in $A$
  - using a sigma-type $\{x \mid A x\}$ with the weak topology would be at best cumbersome
Applications of the Lebesgue Differentiation theorem

▶ FTC (reminder):
For \( f \in L^1 \), \( F(x) \triangleq \int_{t \in [-\infty, x]} f(t)(d\lambda) \) is derivable and \( F'(x) \text{ a.e.} = f(x) \):

Lemma FTC \( f : \mu.-\text{integrable setT (EFin \ o f)} \to \)
let \( F x := (\int[\mu](t \in \]-\infty, x]) (f t))%R in
forall x, lebesgue_pt f x ->
derivable (F : R^o -> R^o) x 1 /
(F : R -> R^o)^\(\sim\) (()) x = f x.

▶ Lebesgue density theorem:
Given \( A \) measurable, \( \lim_{r \to 0^+} \frac{\mu(A \cap B(x, r))}{\mu(B(x, r))} \) is 0 or 1 a.e.:

Lemma density A : measurable A ->
{ae mu, forall x,
mu (A `&` ball x r) * (fine (mu (ball x r)))^-1%:E
@[r --> 0^-'] +] --> (\1_A x)%:E}. 
Related work

- **In CoQ**
  - FTC for the Riemann integral: in CoRN (constructive) [Cru02], CoQ standard library (classical)

- **In NASAlib**
  - no first FTC for Lebesgue integration but an elementary proof (for a $C^1$ function) of the second FTC [NAS23a]

- **in Isabelle/HOL:**
  - first FTC for continuous functions [AHS17, Sect. 3.7]

- **in Lean:**
  - several variants of the first FTC (yet, different hypos/goals)
  - lemma similar to the LDT strengthened with nicely shrinking sets
  - Lebesgue’s density theorem [Nas23b] (using the LDT)
Summary

We brought to CoQ:

▸ the first FTC for Lebesgue integration using the Lebesgue Differentiation theorem

▸ the formal proof is decomposed in standard lemmas

▸ other MathComp-Analysis improvements (lim sup / lim inf, semicontinuity, . . .)

▸ there is even new mathematics inside
  ▸ new proof of Urysohn’s lemma by Zachary

Please consider using MathComp-Analysis version 1.0.0!
Current work

Towards the second FTC for Lebesgue integration

- First FTC for Lebesgue integration ✓
- Lebesgue differentiation thm [IA24] ✓
- Lebesgue’s density thm ✓
- Radon-Nikodým thm [IA24] ✓
- Lebesgue-Stieltjes measure [IA24] ✓
- Theory of bounded/total variation ✓
- Theory of absolutely continuous functions
- Second FTC for Lebesgue integration


