Towards the Fundamental Theorem of Calculus for the Lebesgue Integral in Coq

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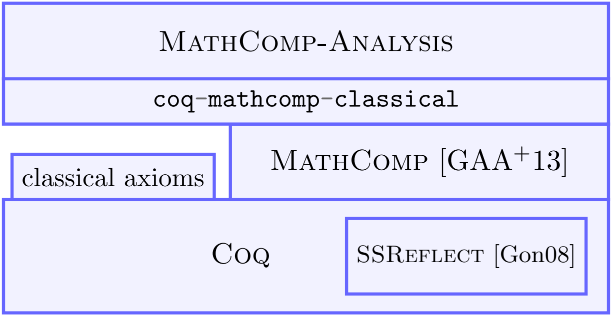
Background

Dependencies of MathComp-Analysis



Motivation

(besides advertising MathComp-Analysis. . .)

The MathComp-Analysis library:

- started with asymptotic reasoning
  - this led to a theory of derivatives [ACR18]
- extended with Lebesgue integration [AC23]
- sample applications:
  - formalization of quantum programs [ZBS+23]
  - formalization of probabilistic programs [ACS23, SA23]

⇒ Link derivatives and Lebesgue integration

⇒ Fundamental Theorem of Calculus for Lebesgue integration

The first FTC for Lebesgue integration

Statement:

- For an integrable function f, define $F(x) \overset{\triangle}{=} \int_{-\infty}^{x} f(t)\mathbf{d}t$.

  $$\text{Then} \quad F \text{ is derivable} \quad \text{and} \quad F'(x) \overset{\text{a.e.}}{=} f(x).$$

Proofs:

- Using theorems already in MathComp-Analysis
  (the dominated convergence theorem, Fatou's lemma, etc., see [AC23])
- ✓ As a consequence of the Lebesgue Differentiation theorem
  - whose proof requires formalization of new standard lemmas
  - which has other applications in itself

Lebesgue Differentiation theorem

Statement

Average of f over A:

$$[f]_A \triangleq \frac{1}{\mu(A)} \int_{y \in A} |f(y)| (\mathbf{d}\, \mu)$$

```
Definition iavg f A :=
 (fine (mu A))^-1%:E *
 \int[mu]_(y in A) `| (f y)%:E |.
```

Deviation of f over $B(x,r)$:

$$\overline{f_{B(x,r)}} \triangleq [\lambda y. f(y) - f(x)]_{B(x,r)}$$

```
Definition favg f x r :=
 iavg (center (f x) \o f)
      (ball x r).
```

Lebesgue point of f at x:

$$\overline{f_{B(x,r)}} \xrightarrow[r \to 0^+]{} 0$$

```
Definition lebesgue_pt f x :=
 favg f x r @[r --> 0^'+] --> 0.
```

Lebesgue differentiation thm:
when f is locally-integrable, we have Lebesgue points a.e.

```
Lemma lebesgue_differentiation f :
 locally_integrable setT f ->
 {ae mu, forall x, lebesgue_pt f x}.
```

Lebesgue Differentiation theorem

Problem reduction

```
Lemma lebesgue_differentiation f :
  locally_integrable setT f ->
  {ae mu, forall x, lebesgue_pt f x}.
```

$$\downarrow$$

Reduce the problem to

$f_k \triangleq f\mathbb{1}_{B_k}$ with $B_k \triangleq B(0, 2(k+1))$ [Sch97, (5.12.101)]

$$\downarrow$$

```
Lemma lebesgue_differentiation_bounded f :
  let B k := ball 0 k.+1.*2%:R in
  let f_ k := f \* \1_(B k) in
  (forall k, mu.-integrable setT (EFin \o f_ k)) ->
  forall k, {ae mu, forall x, x \in B k -> lebesgue_pt (f_ k) x}.
```

Lebesgue Differentiation theorem

```
Lemma lebesgue_differentiation_bounded (f : R -> R) :
  let B k := ball 0 k.+1.*2%:R in
  let f_ k := f \* \1_(B k) in
  (forall k, mu.-integrable setT (EFin \o f_ k)) ->
  forall k, {ae mu, forall x, x \in B k -> lebesgue_pt (f_ k) x}.
```

Proof idea:

- Show that $\forall a > 0, \underbrace{B_k \cap \left\{ x \mid a < \limsup_{r \to 0} \overline{f_k}_{B(x,r)} \right\}}_{**}$ is negligible

  ...

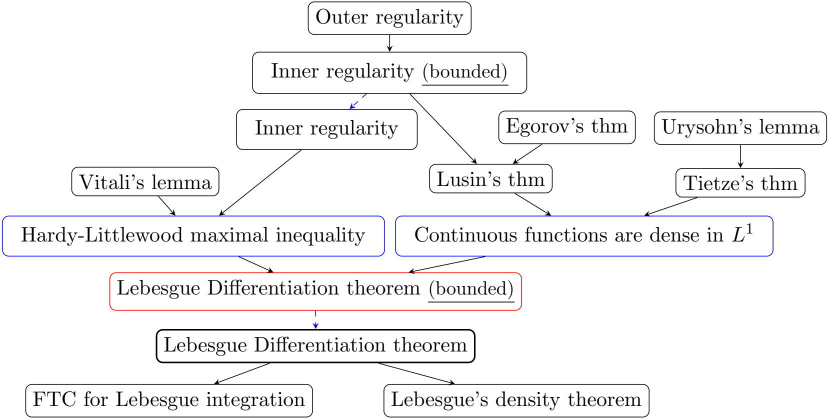
- ... by exhibiting continuous functions g_i such that

$$** \subseteq \bigcap_n B_k \cap \Big(\underbrace{\{x \mid f_k(x) - g_n(x) \geq a/2\}}_{(a)} \cup$$
$$\underbrace{\{x \mid \mathrm{HL}(f_k(x) - g_n(x)) > a/2\}}_{(b)} \Big)$$

$(a) \rightarrow$ Markov's inequality + "continuous functions dense in L^1"

$(b) \rightarrow$ Hardy-Littlewood max. ineq. ($\mathrm{HL}(f)(x) \triangleq \sup_{r>0}\{[f]_{B(x,r)}\}$)

Lebesgue Differentiation theorem: proof organization



Sample lemma: Vitali's covering lemma (finite case)

```
Context {I : eqType}.
Variable B : I -> set R.
Hypothesis is_ballB : forall i, is_ball (B i).
Hypothesis B_set0 : forall i, B i !=set0.

Lemma vitali_lemma_finite s :
  { D | [/\
    {subset D <= s},
    trivIset [set` D] B &
    forall i, i \in s -> exists j,
    [/\ j \in D,
        B i `&` B j !=set0,
        radius (B j) >= radius (B i) &
        B i `<=` 3 *` B j] ] }.
```

Formalization notes:

- When `is_ball A`, the set `A` has a center-point and a `radius`. Since `A` is `set`, a closed ball can be written `closure (B i)`.
- Generalizations in MathComp-Analysis: the infinite case of Vitali's lemma and Vitali's theorem

Sample lemma: Tietze's extension theorem

Given a normal space `X`
and a closed set `A`,
a function `f` continuous on `A`
can be extended
to a function `g` continuous
on the whole set
while preserving boundedness.

```
Context {X : topologicalType}
 {R : realType} (A : set X).
Hypothesis normalX : normal_space X.
Hypothesis clA : closed A.

Lemma continuous_bounded_extension f M :
  0 < M -> {within A, continuous f} ->
  (forall x, A x -> `|f x| <= M) ->
  exists g, [/\ {in A, f =1 g},
                continuous g &
                forall x, `|g x| <= M].
```

- `{within A, continuous f}` states the continuity of `f` with a subspace topology
  - we can write `f (x + eps)`,
    still continuity only depends on the values in `A`
  - using a sigma-type `{x | A x}` with the weak topology would be at best cumbersome

Applications of the Lebesgue Differentiation theorem

- **FTC**(reminder):

  For $f \in L^1$, $F(x) \triangleq \int_{t\in]-\infty,x]} f(t)(\mathbf{d}\,\lambda)$ is derivable and $F'(x) \overset{\text{a.e.}}{=} f(x)$:

```
Lemma FTC f : mu.-integrable setT (EFin \o f) ->
  let F x := (\int[mu]_(t in `]-oo, x]) (f t))%R in
  forall x, lebesgue_pt f x ->
  derivable (F : R^o -> R^o) x 1 /\
  (F : R -> R^o)^`() x = f x.
```

- **Lebesgue density theorem**:

  Given A measurable, $\displaystyle\lim_{r\to 0^+} \frac{\mu(A \cap B(x,r))}{\mu(B(x,r))}$ is 0 or 1 a.e.:

```
Lemma density A : measurable A ->
  {ae mu, forall x,
    mu (A `&` ball x r) * (fine (mu (ball x r)))^-1%:E
      @[r --> 0^'+] --> (\1_A x)%:E}.
```

Related work

- In Coq
  - FTC for the Riemann integral: in CoRN (constructive) [Cru02], Coq standard library (classical)
- In NASAlib
  - no first FTC for Lebesgue integration but an elementary proof (for a C^1 function) of the second FTC [NAS23a]
- in Isabelle/HOL:
  - first FTC for continuous functions [AHS17, Sect. 3.7]
- in Lean:
  - several variants of the first FTC (yet, different hypos/goals)
  - lemma similar to the LDT strengthened with nicely shrinking sets
  - Lebesgue's density theorem [Nas23b] (using the LDT)

Summary

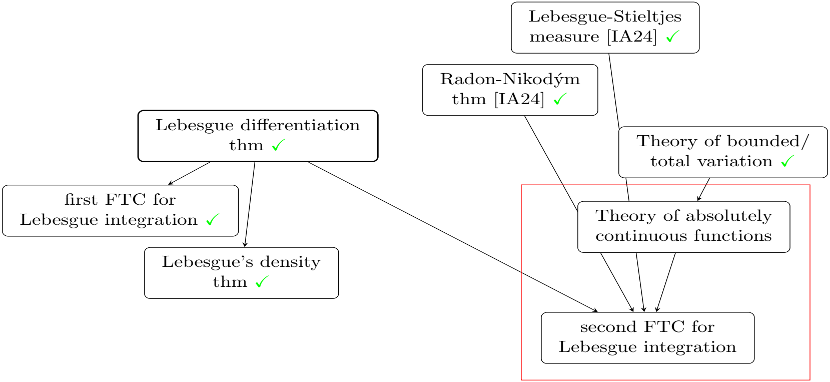
We brought to Coq:

- the first FTC for Lebesgue integration using the Lebesgue Differentiation theorem
- the formal proof is decomposed in standard lemmas
- other MathComp-Analysis improvements ($\limsup$ / $\liminf$, semicontinuity, ...)
- there is even new mathematics inside
  - new proof of Urysohn's lemma by Zachary

Please consider using MathComp-Analysis version 1.0.0!

Current work

Towards the second FTC for Lebesgue integration



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