Verifying "Simple" Persistent Catenable Deques

Journées Francophones des Langages Applicatifs 2026

Juliette Ponsonnet[1] François Pottier[2]

[1]ENS de Lyon

[2]Inria Paris

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1. Introduction
2. The KOT Data Structure
3. Specification
4. Stable References
5. Correctness Proof
6. Time Complexity: Statement and Proof
7. Conclusion

Kaplan, Okasaki and Tarjan's Deques

- catenable deque
- self adjusting data structure
- persistent (old copies cannot be invalidated)
- use references internally
- not so simple
- **all** operations in amortised $O(1)$

Figure: Кот, не очередь

An Unusual Algorithm

Kaplan, Okasaki, Tarjan (2000)

our method requires an extension of the concept of memoization: we allow any expression to be replaced by an equivalent expression

A Surprising Journey

What we ended up doing:

What we set out to do:

- Understand and formalise the extension of memoisation
- Prove the correctness of the data structure
- Prove $O(1)$ amortised time complexity

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- Prove concurrent correctness

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- Prove concurrent correctness
- Formalise sequential "stable references"
- Point out a flaw in the original proof

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- Prove $O(1)$ *sequential* amortised complexity

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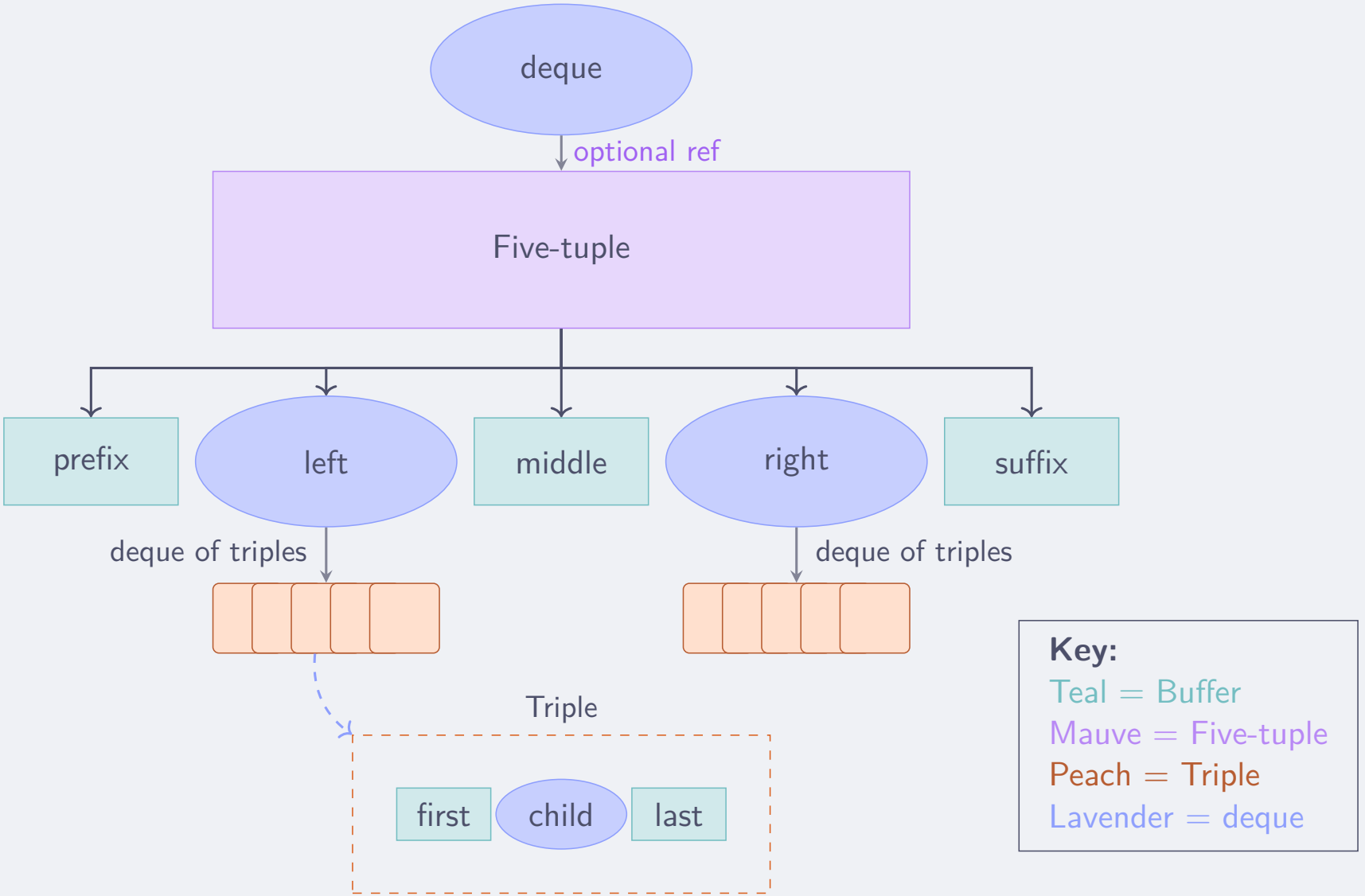
OCaml Interface[1]

```
type 'a deque
val empty : 'a deque
val push  : 'a -> 'a deque -> 'a deque
val inject: 'a deque -> 'a -> 'a deque
val pop   : 'a deque -> 'a * 'a deque
val eject : 'a deque -> 'a deque * 'a
val concat: 'a deque -> 'a deque -> 'a deque
```

[1] opam install kot

Structure of a Deque



OCaml Type Definition

```
type 'a deque =
  'a nonempty_deque option
and 'a nonempty_deque =
  'a five_tuple ref
and 'a five_tuple = {
  prefix : 'a buffer;
  left   : 'a triple deque;
  middle : 'a buffer;
  right  : 'a triple deque;
  suffix : 'a buffer;
}
and 'a triple = {
  first  : 'a buffer;
  child  : 'a triple deque;
  last   : 'a buffer;
}
```

Pop is not so “Simple”

Pop is not so “Simple”

```
let naive_pop_safe (type a) (f : a five_tuple) : bool =
  let { prefix; middle; _ } = f in
  B.is_empty middle || B.length prefix > 3
```

Pop is not so “Simple”

```
let naive_pop_safe (type a) (f : a five_tuple) : bool =
  let { prefix; middle; _ } = f in
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let naive_pop (type a) (f : a five_tuple) : a * a deque =
  (* omitted; just 7 lines *)
```

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let naive_pop_safe (type a) (f : a five_tuple) : bool =
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let naive_pop (type a) (f : a five_tuple) : a * a deque =
  (* omitted; just 7 lines *)

let rec pop_nonempty : type a. a nonempty_deque -> a * a deque = fun r ->
  let f = !r in
  if naive_pop_safe f then
    naive_pop f
  else
    let f = prepare_pop f in
    r := f;
    assert (naive_pop_safe f);
    naive_pop f
```

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    naive_pop f

and prepare_pop : type a. a five_tuple -> a five_tuple = fun f ->
  (* omitted; about 100 lines; uses [pop_triple] *)
```

Foreshadowing

```
and pop_triple
 : type a. a triple nonempty_deque
         -> a triple * a triple deque
 = fun r ->
  let f = !r in
  let t = inspect_first f in
  let { first; last; _ } = t in
  (* The (repaired) mysterious condition: *)
  if not (B.is_empty last) || B.has_length_3 first then
    (* [naive_pop_safe f] is not necessarily true here! *)
    naive_pop f
  else
    pop_nonempty r
```

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Specification

Deque d xs : *d* is a deque *and* represents the list *xs*

$$\mathit{Deque} : \mathit{Val} \to \mathit{list}\ \mathit{Val} \to \mathit{iProp}$$

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DEQUE-PERSIST
persistent(*Deque d xs*)

DEQUE-EMPTY
Deque empty []

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$$\mathit{Deque} : \mathit{Val} \to \mathit{list}\ \mathit{Val} \to \mathit{iProp}$$

DEQUE-PERSIST
$\mathit{persistent}(\mathit{Deque}\ d\ xs)$

DEQUE-EMPTY
$\mathit{Deque}\ \mathit{empty}\ []$

DEQUE-PUSH
$\{\mathit{Deque}\ d\ xs\}\ \mathit{push}\ x\ d\ (\exists d')\ d'\ \{\mathit{Deque}\ d'\ ([x] +\!\!+ xs)\}$

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DEQUE-POP
$\{\mathit{Deque}\ d\ ([x] +\!\!+ xs)\}\ \mathit{pop}\ d\ (\exists d')\ (x, d')\ \{\mathit{Deque}\ d'\ xs\}$

DEQUE-CONCAT
$$\{\mathit{Deque}\ d\ xs\ *\ \mathit{Deque}\ d'\ xs'\}\ \mathit{concat}\ d\ d'\ (\exists d'')\ d''\ \{\mathit{Deque}\ d''\ (xs +\!\!+ xs')\}$$

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This specification makes deques impossible to invalidate, concurrent and shareable.

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Designing the Deque Predicate

How to encode this reference to make the deque predicate persistent?

```
and 'a nonempty_deque =
  'a five_tuple ref
```

Designing the Deque Predicate

How to encode this reference to make the deque predicate persistent? Recall: *We allow ft to be overwritten with a value ft′ provided ft and ft′ are equivalent in some sense.*

```
and 'a nonempty_deque =
  'a five_tuple ref
```

Stable References – Intuition

Standard Iris

$$\ell \mapsto v$$

too precise, exclusive ownership

- Using points-to, restricting writes with a property
- Reference to a property, value unknown

"Stable Reference" library

$$\ell \Mapsto \phi$$

invariant and shareable (everyone agrees on ϕ)

(Concurrent) Stable References (CSRef)

CSRef-Persist
$\mathit{persistent}(\ell \Rrightarrow \phi)$

CSRef-Alloc
$\{\phi\ v\}\ \mathit{ref}\ v\ (\exists \ell)\ \ell\ \{\ell \Rrightarrow \phi\}$

CSRef-Read
$\mathit{persistent}(\phi) \twoheadrightarrow$
$\{\ell \Rrightarrow \phi\}\ !\ell\ (\exists v)\ v\ \{\phi\ v\}$

CSRef-Write
$\{\ell \Rrightarrow \phi\ *\ \phi\ v\}\ \ell := v\ ()\ \{\}$

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$\{\ell \Mapsto \phi\ *\ \phi\ v\}\ \ell := v\ ()\ \{\}$

We also propose a sequential interface (SSRef), discussed later.

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Defining the Deque Predicate

$$\begin{aligned} \mathit{Deque}\ d\ \mathit{xs} \triangleq\ & \ulcorner d = \mathit{None} \wedge \mathit{xs} = [] \urcorner \vee \\ & \exists \ell.\ \ulcorner d = \mathit{Some}(\ell) \urcorner * \\ & \ell \Mapsto (\lambda \mathit{ft}.\ \mathit{fiveTuple}\ n\ \mathit{ft}\ \mathit{xs}) \end{aligned}$$

$$\begin{aligned} \mathit{fiveTuple}\ n\ \mathit{ft}\ \mathit{xs} \triangleq\ & \exists\ p, l, m, r, s, \mathit{xs}_p, \mathit{xss}_l, \mathit{xs}_m, \mathit{xss}_r, \mathit{xs}_s. \\ & \ulcorner d = (p, l, m, r, s) \wedge \\ & \mathit{config}_5(|\mathit{xs}_p|, |\mathit{xss}_l|, |\mathit{xs}_m|, |\mathit{xss}_r|, |\mathit{xss}_s|) \wedge \\ & \mathit{xs} = \mathit{xs}_p +\!\!+ \mathit{join}(\mathit{xss}_l) +\!\!+ \mathit{xs}_m +\!\!+ \mathit{join}(\mathit{xss}_r) +\!\!+ \mathit{xs}_s \urcorner * \\ & \mathit{buffer}\ p\ \mathit{xss}_p * \\ & \mathit{dequeOfTriples}\ l\ \mathit{xs}_l * \\ & \mathit{buffer}\ m\ \mathit{xs}_m * \\ & \mathit{dequeOfTriples}\ r\ \mathit{xss}_r * \\ & \mathit{buffer}\ s\ \mathit{xs}_s \end{aligned}$$

Proof Excerpt – Creating Stable References

Initial Goal

```
"Hb" : buffer (1) [x] b
------------------------------------□
...
------------------------------------*
WP ref (bempty, InjLV #(), bempty, InjLV #(), b)
   {{ v, WP InjR v {{ v, ψ v }} }}
```

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```
wp_apply (csref_alloc (fiveTuple [x])) as "%ℓ #Hℓ".
```

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```

```
wp_apply (csref_alloc (fiveTuple [x])) as "%ℓ #Hℓ".
```

Subgoals

1. **Prove the initial value is correct:**

   ```
   "Hb" : buffer 1 [x] b
   ----------------------------------------□
   fiveTuple [x] (bempty, InjLV #(), bempty, InjLV #(), b)
   ```

2. **Continue the proof with the new reference ℓ:**

   ```
   "Hℓ" : ℓ ⤇ fiveTuple [x]
   ----------------------------------------□
   ...
   ----------------------------------------*
   WP InjR #ℓ {{ v, ψ v }}
   ```

Proving the Specification

- KOT did not provide a correctness proof
- Correctness of the operations is relatively self-evident
- Showing the invariants are upheld requires a more careful look

Pop is not “Simple” at all

Recall:

```
and pop_triple
 : type a. a triple nonempty_deque
         -> a triple * a triple deque
 = fun r ->
  let f = !r in
  let t = inspect_first f in
  let { first; last; _ } = t in
  (* The (repaired) mysterious condition: *)
  if not (B.is_empty last) || B.has_length_3 first then
    (* [naive_pop_safe f] is not necessarily true here! *)
    naive_pop f
  else
    pop_nonempty r
```

Specifying `pop_triple` – Naïve-Pop

NaivePop-Safe

$$\{\mathit{safeFiveTuple}\ n\ ft\ ([x] +\!\!+ xs)\}$$
$$\mathit{naive\_pop}\ ft$$
$$(\exists d)\ (x,\ d)\ \{\mathit{deque}\ n\ d\ xs\}$$

Specifying `pop_triple` – Naïve-Pop

$$\textsc{NaivePop-Safe}\quad \{\mathit{safeFiveTuple}\ n\ ft\ ([x] +\!\!+ xs)\}\ \mathit{naive\_pop}\ ft\ (\exists d)\ (x,\ d)\ \{\mathit{deque}\ n\ d\ xs\}$$

$$\textsc{NaivePop-Unsafe}\quad \{\mathit{fiveTuple}\ n\ ft\ ([x] +\!\!+ xs)\}\ \mathit{naive\_pop}\ ft\ (\exists d)\ (x,\ d)\ \left\{\begin{array}{c}\mathsf{wp}\\ \forall y,\ \mathit{push}\ y\ d\\ (\exists d')\ d'\ \{\mathit{deque}\ n\ d'\ ([y] +\!\!+ xs)\}\end{array}\right\}$$

Specifying `pop_triple` – The Mysterious Condition

The "mysterious condition" seen before is used to decide which path is executed in `pop_triple`. Depending on whether the triple popped is "special", the post-condition is different.

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PopTriple

$$\{\mathit{dequeOfTriples}\ n\ d\ ([t] +\!\!+ ts)\}$$

$$\mathit{pop\_triple}\ d$$

$$(\exists d')\ (t,\ d')\ \left\{ \begin{array}{c} \mathit{nonSpecialTriple}\ n\ xs\ t\ *\ \mathit{deque}\ n\ d'\ ts \\ \bigvee \\ \mathit{specialTriple}\ n\ xs\ t\ * \\ \forall t',\ \mathsf{wp}\ \mathit{push}\ t'\ d'\ (\exists d'')\ d''\ \{\mathit{deque}\ n\ d''\ ([t'] +\!\!+ ts)\} \end{array} \right\}$$

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New Specification

$$\mathit{persistent}(\mathit{Deque}_\$\ \pi\ d\ xs) \qquad\qquad \mathit{Deque}_\$\ \pi\ \mathit{empty}\ []$$

$$\left\{ \mathit{Deque}_\$\ \pi\ d\ xs\ *\ \lightning^{\pi}\ *\ \$7 \right\}$$

$$\mathit{push}\ x\ d$$

$$(\exists d')\ d'\ \left\{ \mathit{Deque}_\$\ \pi\ d'\ ([x] +\!\!+ xs)\ *\ \lightning^{\pi} \right\}$$

$$\left\{ \mathit{Deque}_\$\ \pi\ d\ ([x] +\!\!+ xs)\ *\ \lightning^{\pi}\ *\ \$171 \right\}$$

$$\mathit{pop}\ d$$

$$(\exists d')\ (x, d')\ \left\{ \mathit{Deque}_\$\ \pi\ d'\ xs\ *\ \lightning^{\pi} \right\}$$

$$\left\{ \mathit{Deque}_\$\ \pi\ d\ xs\ *\ \mathit{Deque}_\$\ \pi\ d'\ xs'\ *\ \lightning^{\pi}\ *\ \$57 \right\}$$

$$\mathit{concat}\ d\ d'$$

$$(\exists d'')\ d''\ \left\{ \mathit{Deque}_\$\ \pi\ d''\ (xs +\!\!+ xs')\ *\ \lightning^{\pi} \right\}$$

New Deque Predicate

$$\mathit{Deque}_{\$}\ \pi\ d\ xs \triangleq \mathit{deque}_{\$}\ \pi\ 0\ d\ xs$$

$$\begin{aligned}\mathit{deque}_{\$}\ \pi\ n\ d\ xs \triangleq\ & \ulcorner d = \mathit{None} \wedge xs = [] \urcorner \vee \\ & \exists\, \ell.\ \ulcorner d = \mathit{Some}(\ell) \urcorner * \\ & \ell \overset{\pi.n}{\Longmapsto} (\lambda ft.\ \mathit{fiveTuple}_{\$}\ \pi\ n\ ft\ xs)\end{aligned}$$

$$\begin{aligned}\mathit{fiveTuple}_{\$}\ \pi\ n\ ft\ xs \triangleq\ & \exists\, p, l, m, r, s, xs_p, xss_l, xs_m, xss_r, xs_s. \\ & \$\mathit{potential}(|xs_p|, |xs_s|) * \\ & \ldots\end{aligned}$$

Potential in the structure: the natural amortised analysis technique in **Iris**$^{\$}$.

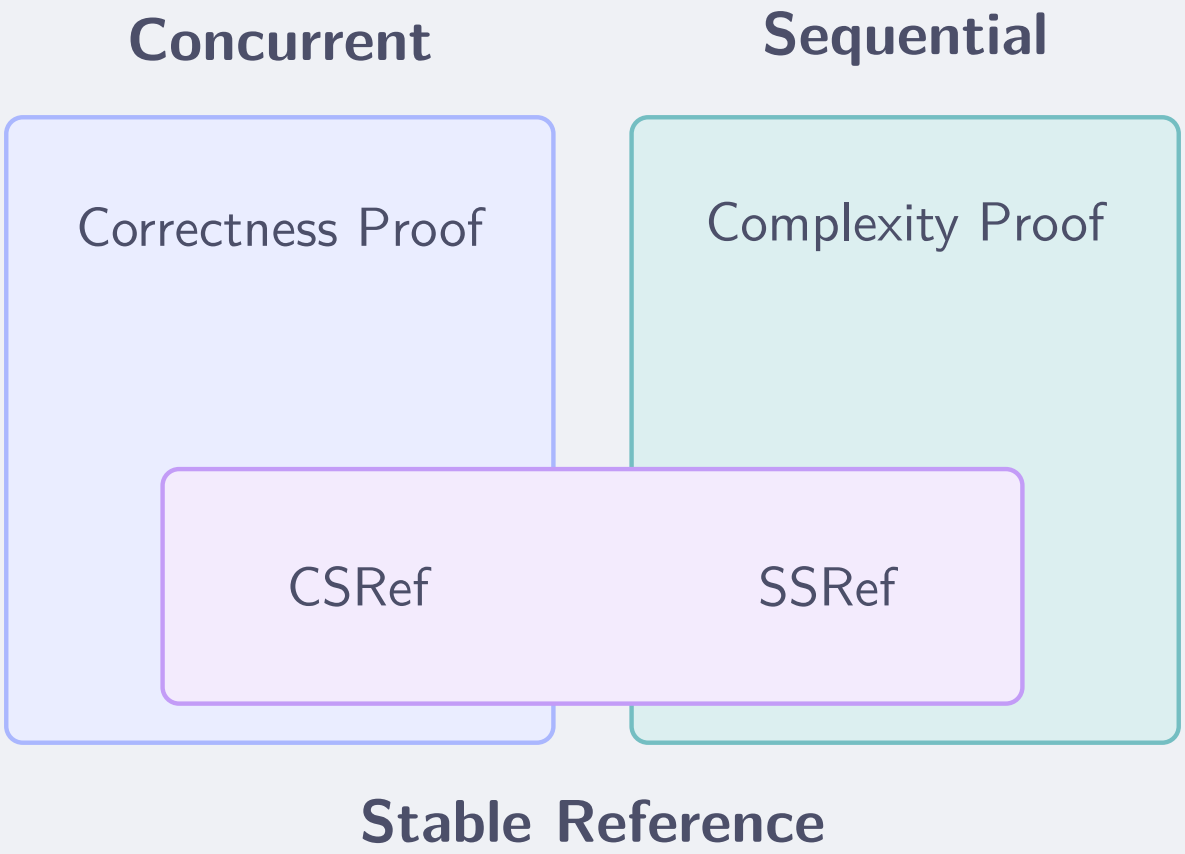
Potential is not Shareable

The concurrent stable references do not work with this definition.

$$\textsc{CSRef-Read}$$
$$\mathit{persistent}(\phi) \mathrel{-\!\!*}$$
$$\{\ell \Mapsto \phi\}\ !\ell\ (\exists v)\ v\ \{\phi\ v\}$$

fiveTuple is not persistent, as it contains credits. We need be able to borrow them and put them back later.

The Need For Two Interfaces



Sequential Stable References (SSRef)

SSRef-New-Pool
$\Rrightarrow \exists \pi.\, \lightning^{\pi.0}$

SSRef-Persist
$\mathit{persistent}(\ell \overset{\pi.n}{\Longmapsto} \phi)$

SSRef-Alloc
$\{\phi\ v\}\ \mathit{ref}\ v\ (\exists \ell)\ \ell\ \{\ell \overset{\pi.n}{\Longmapsto} \phi\}$

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SSRef-Read
$(\forall v.\ \phi\ v \mathrel{-\!\!*} \phi\ v\ *\ \psi\ v) \mathrel{-\!\!*}$
$\{\ell \overset{\pi.n}{\Longmapsto} \phi\ *\ \lightning^{\pi.n}\}\ !\ell\ (\exists v)\ v\ \{\psi\ v\ *\ \lightning^{\pi.n}\}$

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SSRef-Read-Write
$$\left\{\ell \overset{\pi.n}{\Longmapsto} \phi\ *\ \lightning^{\pi.n}\right\}$$
$$!\ell$$
$$(\exists v)\ v\ \{\phi\ v\ *\ \lightning^{\pi.n+1}\ *\ \forall v'.\, \phi\ v' \mathrel{-\!\!*} \lightning^{\pi.n+1} \mathrel{-\!\!*} \mathsf{wp}\ \ell := v'\ ()\ \{\lightning^{\pi.n}\}\}$$

Sequential Stable References (SSRef)

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SSRef-Read-Write
$$\left\{\ell \overset{\pi.n}{\Longmapsto} \phi\ *\ \lightning^{\pi.n}\right\}$$
$$!\ell$$
$$(\exists v)\ v\ \{\phi\ v\ *\ \lightning^{\pi.n+1}\ *\ \forall v'.\, \phi\ v' \mathrel{-\!\!*} \lightning^{\pi.n+1} \mathrel{-\!\!*} \mathsf{wp}\ \ell := v'\ ()\ \{\lightning^{\pi.n}\}\}$$

This interface is not more general than CSRef: it forbids concurrent access.

Proof Excerpt – Temporarily Breaking Invariants

Initial Goal

```
"Hℓ" : ℓ ⤇{π, depth} fiveTuple π n xs
------------------------------------□
"Token" : Token π depth
------------------------------------*
WP let: "f" := ! #ℓ in
   [...]
   #ℓ <- "r";; naive_pop "r"
{{ v, ψ v }}
```

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   #ℓ <- "r";; naive_pop "r"
{{ v, ψ v }}
```

```
wp_apply (ssref_read_write with "[Hℓ Token]") as "%ft (Token & Hft & DONE)".
```

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```
wp_apply (ssref_read_write with "[Hℓ Token]") as "%ft (Token & Hft & DONE)".
```

Subgoals

1. **Provide the token and the reference (`iFrame`).**
2. **Continue the proof:**

```
ft : val
"Hft" : fiveTuple π n xs ft
"Token" : Token π (depth + 1)
"DONE" : ∀t', fiveTuple π n xs t' -∗ Token π (S depth)
                -∗ WP (#ℓ ← t') {{ _, Token π depth }}
------------------------------------*
WP [...]
   #ℓ <- "r";; naive_pop "r"
{{ v, ψ v }}
```

Where the Original Proof Breaks

```
and pop_triple
 : type a. a triple nonempty_deque
         -> a triple * a triple deque
 = fun r ->
  let f = !r in
  let t = inspect_first f in
  let { first; child; last } = t in
  (* The (old)       mysterious condition: *)
  if not   (is_empty child) || B.has_length_3 first then
  (* The (repaired) mysterious condition: *)
  if not (B.is_empty  last) || B.has_length_3 first then
    naive_pop f
  else
    pop_nonempty r
```

The old one breaks potential invariants down the line in `pop_nonempty` in one single possible configuration of `t`. We have checked that this case is feasible.

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Conclusion

Summary of Achievements

- Applied the structure to a concurrent setting.
- Concurrent correctness and $O(1)$ amortised sequential complexity.
- Fixed a flaw in KOT's proof: thanks Rocq!

Key Contribution: Stable References

- We introduced *Stable References*, a novel abstraction for reasoning about shared memory cells supporting restricted updates.
- We developed two distinct interfaces with different capabilities:
  - **CSRef** for fully concurrent read/writes, requires persistent properties.
  - **SSRef** for any property (*e.g* time credits), forbids concurrent access.

Кошки

Figure: Philosophers' Problem

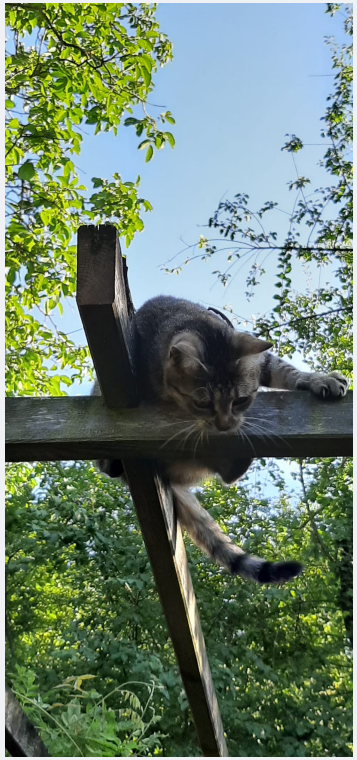

Figure: High Potential Cat

Tokens and Non-Atomic Invariants

Here is the (simplified) API for non-atomic invariants:

NAInv-New-Pool
$\Rrightarrow_0 \exists \pi.\ \lightning^{\pi.0}$

NAInv-New-Inv
$\triangleright P \Rrightarrow_n \square \mathit{NaInv}^{\pi.n}(P)$

NAInv-Acc
$\mathit{NaInv}^{\pi.n}(P) \vdash \lightning^{\pi.n} \propto_n (\triangleright P \ * \ \lightning^{\pi.n+1})$

Where $P \propto_n Q$ means we can get Q from P then recover P from Q[2].

[2] also does view-shifting into the next namespace

Time Credits

$$\textsc{tick\_cost} \in \{0, 1\}$$

Time-Zero
$\$0 \dashv\vdash \mathit{emp}$

Time-Combine
$\$(n + m) \dashv\vdash \$n \ * \ \$m$

Tick-Spec
$\{\$\,\textsc{tick\_cost}\}\ \mathit{tick}()\ ()\ \{\mathit{emp}\}$

OCaml → Tests & Proofs

OCaml Testing

- `Monolith` to identify issues.
- Confirmed KOT error feasibility.

HeapLang for Proof

- Manual translation from OCaml.
- Some recent tools propose automation.

Sequential Specification For `pop_triple`

PopTriple

$$\{\mathit{dequeOfTriples}_{\$}\ \pi\ n\ d\ ([t] +\!\!+ ts)\ *\ \lightning^{\pi \cdot n}\ *\ \$171\}$$
$$\mathit{pop\_triple}\ d$$
$$(\exists d')\ (t,\ d')\ \left\{ \lightning^{\pi \cdot n}\ *\ \left(\begin{array}{c} \mathit{nonSpecialTriple}\ \pi\ n\ xst\ *\ \mathit{deque}_{\$}\ \pi\ n\ d'\ ts \\ \bigvee \\ \mathit{specialTriple}\ \pi\ n\ xst\ *\ \$(171-4)\ * \\ \forall t',\ \mathsf{wp}\ \mathit{push}\ t'\ d'\ (\exists d'')\ d''\ \{\mathit{deque}_{\$}\ \pi\ n\ d''\ ([t'] +\!\!+ ts)\} \end{array} \right) \right\}$$